Math 407 Final Exam Solutions May 10, 2011

Every question is worth 30 points.

1. Use the definition of $\cos z$ to answer the following questions.

   a. Find all zeros of $\cos z$, and their orders.

$$\begin{aligned} \cos z &= \frac{e^{iz}+e^{-iz}}{2} = 0 \implies e^{iz} = -e^{-iz} \implies e^{2iz} = -1 = e^{\pi i + 2k\pi i} \\ 2iz &= \pi i + 2k\pi i \implies \\ z &= \frac{\pi}{2} + k\pi = (2k+1)\frac{\pi}{2}, \text{ where } k \text{ is any integer.} \end{aligned}$$

   Since the derivative of $\cos z$ is $-\sin z$ and $\sin((2k+1)\frac{\pi}{2}) \neq 0$, these are zeros of order 1.

   b. Find all solutions, if any, of the equation $\cos z = 2$.

$$\begin{aligned} \cos z &= \frac{e^{iz}+e^{-iz}}{2} = 2 \implies e^{2iz} - 4e^{iz} + 1 = 0 \implies e^{iz} = 2 \pm (3)^{1/2} \\ iz &= \log(2 \pm (3)^{1/2}) = \ln|2 \pm (3)^{1/2}| + i \arg(2 \pm (3)^{1/2}) \\ &= \ln|2 \pm (3)^{1/2}| + i2k\pi \implies \\ z &= 2k\pi - i \ln|2 \pm (3)^{1/2}| = 2k\pi \pm i \ln(2 + (3)^{1/2}), \text{ where } k \text{ is any integer.} \end{aligned}$$

   Note: $\ln(2-\sqrt{3}) = -\ln(2+\sqrt{3})$. So the solutions of $\cos z = 2$ are symmetric with respect to the real axis.

2. Suppose the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{z^n}$ converges at a point $z_0 \neq 0$. For what other values of z do we know for sure that this series converges?

   The series will converge for all z such that $|z| > |z_0|$. Since the series converges at z_0 we must have

$$\overline{\lim}_{n\to\infty} \left| \frac{a_{n+1}/z_0^{n+1}}{a_n/z_0^n} \right| = \frac{1}{|z_0|} \overline{\lim_{n\to\infty}} \left| \frac{a_{n+1}}{a_n} \right| \leq 1\,.$$

   So if $|z| > |z_0|$ we have

$$\overline{\lim_{n\to\infty}} \left| \frac{a_{n+1}/z^{n+1}}{a_n/z^n} \right| = \frac{1}{|z|} \overline{\lim_{n\to\infty}} \left| \frac{a_{n+1}}{a_n} \right| < \frac{1}{|z_0|} \overline{\lim_{n\to\infty}} \left| \frac{a_{n+1}}{a_n} \right| \leq 1$$

   and the series will converge.

   Note that the use of $<$ in the above line assumes that $\overline{\lim}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is positive. If it is zero then there is nothing to prove, and the series will converge for all $z \neq 0$.

3. Define that branch of $\log z$ for which $\log(-1) = i\pi$ in terms of $\arctan \frac{y}{x}$ for $x > 0$. Then show that $\log z$ is an analytic function of z and $\frac{d \log z}{dz} = \frac{1}{z}$. Note, I am not asking you to define $\log z$ or show that it is analytic for all $z \neq 0$, just those z for which $\mathrm{Re}(z) > 0$.

   For any z which has positive real part define $\log z$ by

$$\log z = \ln|z| + i \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right), \text{ where } z = x + iy\,.$$

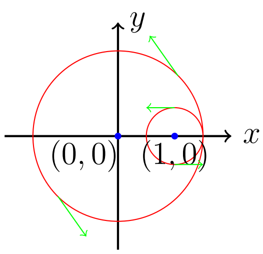
   Checking to see that the Cauchy-Riemann equations are satisfied we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \\
\frac{\partial v}{\partial x} &= \frac{1}{1+(y^2/x^2)}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = -\frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial y} &= \frac{1}{1+(y^2/x^2)}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x}
\end{aligned}$$

   Since the functions u and v have continuous partial derivatives $(x > 0)$ the sufficient Cauchy-Riemann equations are satisfied. The derivative of log equals

$$\frac{d\log z}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}\,.$$

4. Let Γ be the curve consisting of the two circles $|z-1| = \frac{1}{2}$ and $|z| = \frac{3}{2}$. See the plot below.



   The outer circle being traversed first, starting at the point $(3/2, 0)$, in a counter clockwise direction, and then, after returning to the starting point, the inner circle is traversed also in a counter clockwise direction. Let C_1 denote the outer circle, and C_2 denote the inner circle. Then we can think of Γ as $C_1 + C_2$.

   a. Let $f(z) = \dfrac{1}{\sin(\pi z)}$. Where are the singularities of $f(z)$ located, and what kind of singularities are they?

      The function $g(z) = \sin \pi z$ is entire, which means its reciprical is analytic except where $g(z) = 0$, which occurs only when z is any integer. Moreover these are zeros of order 1. Thus, $f(z)$ has poles of order 1 at the points $z = k$ for any integer k.

b. Calculate the value of $\frac{1}{2\pi i}\int_\Gamma \frac{1}{\sin(\pi z)}\,dz$.

We note that the only places the integrand has a singularity inside the curve Γ are at the points $z=-1$, $z=0$, and $z=1$.

$$
\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{1}{\sin(\pi z)}\,dz &= \frac{1}{2\pi i}\int_{C_1} \frac{1}{\sin(\pi z)}\,dz + \frac{1}{2\pi i}\int_{C_2} \frac{1}{\sin(\pi z)}\,dz \\
&= [\mathrm{Res}(f,-1)+\mathrm{Res}(f,0)+\mathrm{Res}(f,1)] + \mathrm{Res}(f,1) \\
&= \mathrm{Res}(f,-1)+\mathrm{Res}(f,0)+2\mathrm{Res}(f,1) \\
&= \lim_{z\to -1}\frac{z+1}{\sin \pi z} + \lim_{z\to 0}\frac{z}{\sin \pi z} + 2\lim_{z\to 1}\frac{z-1}{\sin \pi z} \\
&= \frac{-1}{\pi} + \frac{1}{\pi} + 2\frac{-1}{\pi} = \frac{-2}{\pi}\,.
\end{aligned}
$$

5. Let C_R denote that part of the circle of radius R centered at the origin, which lies above the real axis. Show that

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z}\,dz = 0\,.$$

$$
\begin{aligned}
\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right| &= \left|\int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{z} iz d\theta\right| \le \int_0^\pi e^{-R\sin\theta}d\theta \\
&\le \int_0^\delta d\theta + \int_\delta^{\pi-\delta} e^{-R\sin\delta}d\theta + \int_{\pi-\delta}^{\pi} d\theta \\
&= 2\delta + (\pi-2\delta)e^{-R\sin\delta} \text{ for any } \delta \text{ which satisfies } 0<\delta<\frac{\pi}{2}\,.
\end{aligned}
$$

Since $-R\sin\delta$ goes to $-\infty$ as R goes to ∞ we see that the limiting value as $R\to\infty$ of the absolute value of the integral can be made as small as we went. Hence, this limiting value must be zero.