1. (20) Define or state the following

(a) f is uniformly continuous on the interval I.

A function f is uniformly continuous on a set I if for any $\epsilon > 0$, there is a $\delta > 0$ such that if x and y are in I and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

(b) f is Riemann integrable on the interval $[a, b]$.

A function f is Riemann integrable on a closed bounded interval if f is bounded on the interval, and if for any $\epsilon > 0$ there is a partition P of the interval such that

$$\mathrm{U}(f, P) - \mathrm{L}(f, P) < \epsilon,$$

where $\mathrm{U}(f, P)$ and $\mathrm{L}(f, P)$ are the upper and lower Riemann sums respectively.

(c) State the mean value theorem.

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a point $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

(d) What does it mean to say that f is differentiable at the point x_0?

This means that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

2. (10) Let $[a, b]$ be a closed and bounded interval. Suppose that $f \in C[a, b]$, and $f(x) < 0$ for all $x \in [a, b]$. Show there is an $\alpha < 0$ such that $f(x) \leq \alpha$ for all $x \in [a, b]$.

Since f is continuous on the closed bounded interval $[a, b]$, there is a point $x_{\text{sup}} \in [a, b]$ at which f attains its supremum. That is, $f(x_{\text{sup}}) \geq f(x)$ for all $x \in [a, b]$. Set $\alpha = f(x_{\text{sup}})$. Then, since $f(x) < 0$ for all $x \in [a, b]$ we have $\alpha < 0$, and

$$f(x) \leq f(x_{\text{sup}}) = \alpha < 0.$$

3. (10) Use the definition of the derivative to find the derivative of $f(x) = x^2$ at the point $x = -2$.

$$\begin{aligned} f'(-2) &= \lim_{h \to 0} \frac{(-2 + h)^2 - (-2)^2}{h} = \lim_{h \to 0} \frac{4 - 4h + h^2 - 4}{h} \\ &= \lim_{h \to 0} (-4 + h) \\ &= -4 . \end{aligned}$$

4. (10) Suppose that f is uniformly continuous on the interval $[a,b]$. Show that if the sequence $x_n \in [a,b]$ is a Cauchy sequence, then $f(x_n)$ is also a Cauchy sequence.

Let $\epsilon > 0$. Then, since f is uniformly continuous on $[a,b]$, there is a $\delta > 0$ such that if x and y belong to $[a,b]$, and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since the squence x_n is Cauchy there is an N such that if n and m are both greater than N we have $|x_n - x_m| < \delta$. Thus, if n, $m > N$ we must have

$$|f(x_n) - f(x_m)| < \epsilon.$$

Thus, the sequence $f(x_n)$ is Cauchy.

5. (10) Let $f(x) = \sqrt{1+2x}$ and $g(x) = 1 + x$. Note the domain of f is the interval $[-1/2, \infty)$, and the domain of g is the set of all real numbers. Find all x in the domain of f, such that $f(x) \leq g(x)$.

It turns out that $f(x) \leq g(x)$ for all $x \in [-1/2, \infty)$. To see this set

$$\begin{aligned} h(x) &= g(x) - f(x) \\ &= 1 + x - \sqrt{1+2x}. \end{aligned}$$

Taking the derivative of h we have

$$h'(x) = 1 - \frac{1}{\sqrt{1+2x}} .$$

Then it is easy to see that for $x \in [-1/2, 0)$, $h'(x) < 0$, $h'(0) = 0$, and for $x \in (0, \infty)$, $h'(x) > 0$. Thus, h is decreasing on the interval $[-1/2, 0]$ and increasing on the interval $[0, \infty)$. This means that $h(x) \geq h(0) = 0$ for all $x \in [-1/2, \infty)$. But $h \geq 0$ is the same as $g \geq f$.
Another way to see this is to use the Mean Value theorem. Then

$$\begin{aligned} h(x) &= h(x) - h(0) \\ &= h'(\xi)(x - 0), \end{aligned}$$

where ξ lies between x and 0. Thus, if $x \in [-1/2, 0)$, then $x < 0$ and $h'(\xi) < 0$, which implies $h(x) > 0$. If $x \in (0, \infty)$, then both x and $h'(\xi)$ are postive and we again have $h(x) > 0$. The only value of x for which $h(x) = 0$ is at $x = 0$. Or the only value of x for which $g(x)$ is not greater than $f(x)$ is at $x = 0$.

6. (15) Let $f(x) = \begin{cases} 1 & -1 \leq x < 0 \\ 3 & x = 0 \\ 2 & 0 < x \leq 4 \end{cases}$.

(a) Let $P = \{-1, -1/2, 0, 4\}$. Compute $U(f,P)$ and $L(f,P)$.

$$\begin{aligned} \mathrm{U}(f,P) &= M_1 \frac{1}{2} + M_2 \frac{1}{2} + M_3(4) = \frac{1}{2} + \frac{3}{2} + 12 = 14 \\ \mathrm{L}(f,P) &= m_1 \frac{1}{2} + m_2 \frac{1}{2} + m_3(4) == \frac{1}{2} + \frac{1}{2} + 8 = 9. \end{aligned}$$

(b) Show that $f \in R[a,b]$.

Let $P = \{-1, -\delta, \delta, 4\}$. Then we have

$$\begin{aligned} \mathrm{U}(f,P) - \mathrm{L}(f,P) &= 0(1-\delta) + (3-1)(2\delta) + 0(4-\delta) \\ &= 4\delta\ . \end{aligned}$$

So, given any $\epsilon > 0$, set $\delta = \min(1, \epsilon/4)$ (don't want $-\delta < -1$ or $\delta > 4$). Then for this δ the difference of the upper and lower Riemann sums is less than ϵ.

7. (15) Let $F(x) = \int_0^x f(t)\ dt$, where f is the function from problem 6.

(a) What is the domain of F ?

The domain of F is the same as the domain of f. That is, $[-1,4]$.

(b) What is $F(-1)$?

$$\begin{aligned} F(-1) &= \int_0^{-1} f(t)\ dt = -\int_{-1}^{0} f(t)\ dt \\ &= -1. \end{aligned}$$

(c) Show that F is continuous on its domain.

For any x,$y \in [-1,4]$, we have

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_0^x f(t)\ dt - \int_0^y f(t)\ dt \right| \\ &= \left| \int_y^x f(t)\ dt \right| \leq \int_{\min\{x,y\}}^{\max\{x,y\}} |f(t)|\ dt \\ &\leq \int_{\min\{x,y\}}^{\max\{x,y\}} 3\ dt = 3|x-y|\ . \end{aligned}$$

Thus, we see that not only is F continuous on $[-1,4]$, but it is also uniformly continuous.

8. (10) Let f be Riemann integrable on the closed bounded interval $[a, b]$. Let $M = \sup\{f(x) : a \le x \le b\}$ and $m = \inf\{f(x) : a \le x \le b\}$.

(a) Show there is a number c such that

$$c(b-a) = \int_a^b f(x)\ dx,$$

where $m \le c \le M$.

We clearly have

$$m(b-a) \le \int_a^b f(x)\ dx \le M(b-a).$$

Thus, the ratio $\frac{\int_a^b f(x)\ dx}{b-a}$ lies between m and M. Set c equal to this ratio.

(b) Find such a c for the particular function f of problem 6.

For the function of problem 6 we have

$$c = \frac{\int_a^b f(x)\ dx}{b-a} = \frac{\int_{-1}^4 f(x)\ dx}{5} = \frac{9}{5}$$