1. (10) Let f be a real valued function defined on the closed bounded interval $[a, b]$.

   (a) Define what it means to say that f is differentiable at the point a.

   f is differentiable at a means that the following limit exists

   $$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}.$$

   (b) Let c be any point in the open interval (a, b). Define what it means to say that f is differentiable at c.

   f is differentiable at c means that the following limit exists

   $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

2. (25) State and prove Rolle's theorem.

   Let f be continuous on the closed bounded interval $[a, b]$, and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$, such that $f'(c) = 0$.

   Since f is continuous on $[a, b]$, there exist points x_1 and x_2 in $[a, b]$ where f takes on its supremum and infimum. If neither of these points is in the open interval (a, b), then, since $f(a) = f(b)$, we have $\inf f = \sup f$, which implies that f is constant on $[a, b]$ and we have $f'(x) = 0\ \forall x \in (a, b)$. Now suppose that at least one of the points x_i is in (a, b). Then f has a local extremum at this point, which implies that the derivative of f must be zero at this point.

3. (25) Show that if f is continuous on the closed bounded interval $[a, b]$, then $J = \{ f(x) : a \leq x \leq b \}$ is also a closed bounded interval.

   Let $c = \inf\{ f(x) : x \in [a, b] \}$ and $d = \sup\{ f(x) : x \in [a, b] \}$. Since f is continuous on the closed bounded interval $[a, b]$ there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = c$ and $f(x_2) = d$. Clearly $J \subset [c, d]$.

   Since $c = f(x_1)$ and $d = f(x_2)$ also, we have both c and d in J. Moreover if $y \in (c, d)$, then by the intermediate value theorem for continuous functions we have $y = f(x)$ for some $x \in [a, b]$. Thus, $y \in J$, and $[c, d] \subset J$. Hence $J = [c, d]$ a closed bounded interval.

4. (40) Let f be a real valued bounded function defined on the closed bounded interval $[a, b]$.

(a) Define an upper Riemann sum, U(f, P), of f.

Let $P = \{x_i\}_{i=0}^n$ be any partition of $[a, b]$. Let $M_j = \sup\{ f(x) : x_{j-1} \leq x \leq x_j\}$. Then the upper Riemann sum of f with respect to this partition is

$$U(f, P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1})\,.$$

(b) Let P and Q be any two partitions of $[a, b]$. Show that

$$L(f, P) \leq U(f, Q)\,,$$

where $L(f, P)$ and $U(f, P)$ denote lower and upper Riemann sums of f respectively.

It is true that if Q is any partition of $[a, b]$, which refines a partition P, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq L(f, Q)$. Since for any two partitions P and Q, $P \cup Q$ is a refinement of both P and Q, we have

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)\,.$$

(c) Define what "f is Riemann integrable on $[a, b]$" means. Then define $\int_a^b f(x)\,dx$, the integral from a to b of f.

f is said to be integrable on $[a, b]$ if for any $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon\,.$$

If f is integrable on $[a, b]$ we know that the upper and lower Riemann sums are equal. That is, the numbers

$$\begin{aligned} (L)\int_a^b f(x)\,dx &= \sup\{L(f, P) : P \text{ is any partition of } [a, b]\} \\ (U)\int_a^b f(x)\,dx &= \inf\{L(f, P) : P \text{ is any partition of } [a, b]\} \end{aligned}$$

are equal, and this common value is called the integral of f over the interval $[a, b]$.

(d) Let f be continuous on the closed bounded interval $[a, b]$.

i. Show that f is integrable on $[a, b]$.

Let $\epsilon > 0$ be given. Then, since f is continuous on a closed bounded interval, it is also uniformly continuous on this interval. Let $\delta > 0$ be such that if $|x-y| < \delta$ and both x and y are in $[a, b]$, then $|f(x) - f(y)| < \epsilon/(b-a)$. Let P be any partition of $[a, b]$ such that $||P|| < \delta$. Since f is continuous on each of the subintervals $[x_{j-1}, x_j]$, are are points x_j and y_j in this subinterval such that $f(x_j) = M_j$ and $f(y_j) = m_j$. Moreover $|x_j - y_j| \leq ||P|| < \delta$. Thus, we have

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{j=1}^{n} M_j \Delta x_j - \sum_{j=1}^{n} m_j \Delta x_j = \sum_{j=1}^{n} \left(f(x_j) - f(y_j) \right) \Delta x_j \\ &< \sum_{j=1}^{n} \frac{\epsilon}{b-a} \Delta x_j = \frac{\epsilon}{b-a}(b-a) = \epsilon \end{aligned}$$

ii. Show there is a point x_0 in $[a, b]$ such that

$$f(x_0) = \frac{\int_a^b f(x)\,dx}{b-a}\,.$$

Let $m = \inf\{\, f(x) : x \in [a, b]\}$ and $M = \sup\{\, f(x) : x \in [a, b]\}$. Then we have

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)\,.$$

Thus, the number $\frac{\int_a^b f(x)\,dx}{b-a}$ lies between m and M. The intermediate value theorem for continuous functions guaranties that there is a number $x_0 \in [a, b]$ such that $f(x_0)$ equals this number. That is,

$$f(x_0) = \frac{\int_a^b f(x)\,dx}{b-a}\,.$$