

1. (a) (5) State the completeness axiom for real numbers.

Let E be any set of real numbers, which is bounded above. Then E has a least upper bound, which, since E is bounded, is finite.

- (b) (15) Show that any monotone sequence which is bounded must converge.

With out loss of generality we can assume that our sequence, $\{a_n\}$ is monotone increasing. Since the sequence is bounded it has a least upper bound, which we'll denote by l . Let $\epsilon > 0$. Then there is a number n_ϵ such that

$$l - \epsilon < a_{n_\epsilon} \leq l.$$

Since the sequence a_n is increasing this means that for any $n > n_\epsilon$ we have $l - \epsilon < a_n < l$. Thus, for any $\epsilon > 0$ we have found an $N (= n_\epsilon)$ such that for $n > N$ we have $|a_n - l| < \epsilon$. Thus, the sequence converges to l its least upper bound.

2. (20) Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of real numbers.

- (a) Define what $\limsup_{n \rightarrow \infty} a_n = l$ means.

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup \{a_k : k \geq n\}.$$

- (b) Show that $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$.

Let $\epsilon > 0$. Then there are numbers N_1 and N_2 such that for $n > N_1$ and $m > N_2$ we have

$$a_n < \limsup_{n \rightarrow \infty} a_n + \frac{\epsilon}{2} \text{ and } b_m < \limsup_{n \rightarrow \infty} b_n + \frac{\epsilon}{2}.$$

Thus, for $n > \hat{N} = \max\{N_1, N_2\}$ we have

$$a_n + b_n < \left(\limsup_{n \rightarrow \infty} a_n + \frac{\epsilon}{2} \right) + \left(\limsup_{n \rightarrow \infty} b_n + \frac{\epsilon}{2} \right).$$

Thus, for any $\epsilon > 0$ we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \sup \left\{ a_k + b_k : k \geq \hat{N} + 1 \right\} \leq \left(\limsup_{n \rightarrow \infty} a_n + \frac{\epsilon}{2} \right) + \left(\limsup_{n \rightarrow \infty} b_n + \frac{\epsilon}{2} \right).$$

We can now conclude that $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$.

3. (20) Suppose that f and g are defined on an open interval containing the point a . Suppose in addition that $g(x) \neq 0$ for all x not equal to a . Suppose also that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \text{ and } \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = 1.$$

Show that $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$. I'm asking you to prove this special case of l'Hôpital's rule.

We can assume without loss of generality that both f and g are defined at a and take on the value 0 at this point. Thus, for both functions there is a $\delta > 0$ such that they satisfy the hypotheses of the Generalized Mean Value theorem on the intervals $[a - \delta, a]$ and $[a, a + \delta]$. Thus, for any $x \in [a - \delta, a + \delta] \setminus \{a\}$ we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)},$$

for some ξ between x and a . This, means that as $x \rightarrow a$, we must have ξ also converging to a . Thus,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{g(x) - g(a)} \right] = \lim_{x \rightarrow a} \frac{f'(\xi)}{g'(\xi)} = 1.$$

4. (a) (10) What does it mean to say that $f(x)$ is Riemann integrable on the closed bounded interval $[a, b]$.

For every $\epsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$, where $U(f, P)$ and $L(f, P)$ denote the upper and lower Riemann sums for f associated with the partition P .

- (b) (10) Prove that x^2 is Riemann integrable on $[1, 4]$.

The easiest way is to remark the x^2 is continuous on the interval, and is therefore Riemann integrable on the interval. However, let's actually show how to find a partition for a given ϵ . Using the mean value theorem we see that for any x and y in the interval $[1, 4]$ the following inequality is valid

$$|x^2 - y^2| = |2\xi(x - y)| \leq 8|x - y|.$$

For any $\epsilon > 0$ Let P be any partition whose norm is less than $\frac{\epsilon}{8(4-1)}$. For such a partition we have

$$U(f, P) - L(f, P) = \sum_{k=1}^n (M_k - m_k) \Delta x_k \leq \sum_{k=1}^n 8 \frac{\epsilon}{8(4-1)} \Delta x_k = \frac{\epsilon}{4-1} \sum_{k=1}^n \Delta x_k = \epsilon$$

Thus, the function x^2 is Riemann integrable on the interval $[1, 4]$.

(c) (15) Explain why $\int_1^4 x^2 dx = \frac{4^3}{3} - \frac{1^3}{3}$. It is not sufficient to say that $\frac{d}{dx} \left(\frac{x^3}{3} \right) = x^2$.

We have already seen that x^2 is Riemann integrable on the interval $[1, 4]$, and since it is the derivative of $x^3/3$, the second version of the fundamental theorem of calculus tells us that

$$\int_1^4 x^2 dx = \frac{x^3}{3} \Big|_1^4 = \frac{63}{3} = 21.$$

5. (a) (10) Define what it means to say the series $\sum_{n=1}^{\infty} a_n$ converges.

Let $S_n = \sum_{k=1}^n a_k$. The infinite series $\sum_{n=1}^{\infty} a_n$ is said to converge if

$$\lim_{n \rightarrow \infty} S_n$$

exists and is finite.

(b) (15) State and prove the ratio test for convergence of a positive term series.

Let $a_n > 0$ for each n . Let $L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$. If $L < 1$ the series $\sum_{n=1}^{\infty} a_n$ converges, if $L > 1$ the series diverges, and if $L = 1$ nothing can be said.

Suppose $L < 1$. Let η be any number strictly between L and 1. Then there is an N such that for $n \geq N$ we have $\frac{a_{n+1}}{a_n} < \eta$. This implies the following inequalities:

$$a_{N+1} < a_N \eta, \quad a_{N+2} < a_{N+1} \eta < a_N \eta^2 \quad \text{and finally} \quad a_{N+k} < a_N \eta^k.$$

Using the limit comparison test on the given series and the geometric series η^n , we see that the series $\sum_{n=1}^{\infty} a_n$ must converge.

If $L > 1$, then let η be any number strictly between L and 1. Then there is an N such that for $n \geq N$ we have $\frac{a_{n+1}}{a_n} > \eta$. Thus, for any $k > 0$, we have

$$a_{N+k} > a_N \eta^k > a_N.$$

This inequality tells us that $\lim_{n \rightarrow \infty} a_n \neq 0$, which means the series diverges.

If $L = 1$, look at the p -series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$. The ratio test for any of these series has $L = 1$, but the series converges if $p > 1$ and diverges if $p \leq 1$.

6. (30) Decide whether the following series converge, converge absolutely, or diverge.

$$(a) \sum_{k=1}^{\infty} \frac{k^2}{2^k} \quad (b) \sum_{k=2}^{\infty} \frac{(-1)^k}{k(\ln k)^4} \quad (c) \sum_{k=1}^{\infty} \frac{(-1)^k k^3}{(k+1)!}$$

All three of these series converge absolutely. Use the ratio test on the first and third series, and the integral test on the second. The calculations follow.

(a)

$$\lim_{k \rightarrow \infty} \frac{(k+1)^2/2^{k+1}}{k^2/2^k} = \lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \right)^2 \frac{1}{2} = \frac{1}{2}.$$

(b)

$$\int_2^{\infty} \frac{dx}{x(\ln x)^4} = \int_{\ln 2}^{\infty} \frac{du}{u^4} < \infty.$$

(c)

$$\lim_{k \rightarrow \infty} \frac{(k+1)^3/(k+1)!}{k^3/k!} = \lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \right)^3 \frac{1}{k+1} = 0.$$