

This exam consists of two parts, A and B. There are seven (7) questions in part A. Work all of them. There are three (3) questions in part B. Work only one of them, and be sure to tell me which one you work. The questions should be answered in order. That is, number one is first in your blue book, number two second, etc. The single question from part B, which you answer, should appear last in your blue book.

Part A.

1. (10) State the completeness property of real numbers.

Ans: If A is any set of real numbers which is bounded above, then the set A has a supremum.

2. (15) Let $S = \{x \in \mathbb{R} : x^4 \leq 5\}$. Let $s = \sup(S)$. Show that $s^4 = 5$. Note: for this problem you may not assume any n^{th} root of a number exists. In particular square roots and fourth roots are not known to exist.

Ans: We show that s^4 cannot be less than 5 nor more than 5. Hence, it must equal 5. Suppose first that $s^4 < 5$. Then there is an $\epsilon > 0$ such that $(s + \epsilon)^4 < 5$ too. This contradicts the assumption that s is the supremum of the set of numbers, x , which satisfy $x^4 < 5$. To see that such an ϵ exists expand $(s + \epsilon)^4 < 5$. We want to find a positive ϵ such that

$$\epsilon(4s^3 + 6\epsilon s^2 + 4\epsilon^2 s + \epsilon^3) < 5 - s^4.$$

The Archimedean principle can be used (note the term in parentheses is bounded) to show that such an ϵ exists.

We next assume that $s^4 > 5$. Then there exists a positive number ϵ such that $(s - \epsilon)^4 > 5$. But this also contradicts the fact that s is the supremum of all such numbers. For if $x^4 < 5 < (s - \epsilon)^4$, then we must have $x < s - \epsilon$, and s is supposed to be smaller than any other upper bound of these numbers.

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3. (10) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. State the definition of $\lim_{n \rightarrow \infty} a_n = l$. There are three definitions required here. One for l a real number, and two more for $l = \pm\infty$.

Ans: Suppose that l is a real number. Then $\lim_{n \rightarrow \infty} a_n = l$ means that for any $\epsilon > 0$ there is a number N such that if $n > N$, then $|a_n - l| < \epsilon$.

Suppose $l = \infty$. Then $\lim_{n \rightarrow \infty} a_n = l$ means that for any number M there is an N such that if $n > N$, then $M < a_n$.

Suppose $l = -\infty$. Then $\lim_{n \rightarrow \infty} a_n = l$ means that for any number M there is an N such that if $n > N$, then $M > a_n$.

4. (15) Use the definition of limit to prove that $\lim_{n \rightarrow \infty} \left(\frac{n-1}{2n+1} \right) = \frac{1}{2}$.

Ans: Let $\epsilon > 0$. Pick $N = \frac{3}{\epsilon}$. Then if $n > N$ we have

$$\begin{aligned} \left| \left(\frac{n-1}{2n+1} \right) - \frac{1}{2} \right| &= \left| \frac{3}{2(2n+1)} \right| \\ &\leq \left| \frac{3}{n} \right| \\ &< 3 \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

5. (10) Define $\limsup_{n \rightarrow \infty} a_n$.

Ans: For each $n \in \mathbb{N}$ define $b_n = \sup_{k \geq n} \{a_k\}$. Then the \limsup is defined as

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

6. (15) Let

$$a_n = \begin{cases} n, & 1 \leq n \leq 20 \\ 2, & 21 \leq n \leq 100 \\ \frac{n}{n+1}, & 101 \leq n \end{cases}$$

Find the following:

(a) $\sup_{k \geq 10} \{a_k\}$.

(b) $\limsup_{n \rightarrow \infty} a_n$.

(c) $\liminf_{n \rightarrow \infty} a_n$.

Ans: $\sup_{k \geq 10} \{a_k\} = 20$. $\limsup_{n \rightarrow \infty} a_n = 1$. $\liminf_{n \rightarrow \infty} a_n = 1$.

7. (10) Define a Cauchy sequence.

Ans: A sequence $\{a_n\}$ is called a Cauchy sequence if for every $\epsilon > 0$ there is an N such that if m and n are both greater than N , then $|a_n - a_m| < \epsilon$.

Part B.

8. (15) Show that

$$l = \limsup_{n \rightarrow \infty} a_n,$$

where l is a finite real number, if and only if the following two conditions are satisfied:

- (a) For all $\epsilon > 0$ there exists an N such that if $n > N$ then $a_n < l + \epsilon$.
- (b) For all $\epsilon > 0$ and for all N there exists an $n > N$ such that $a_n > l - \epsilon$.

Ans: Suppose that l is the limsup of the sequence $\{a_n\}$. Let ϵ be any positive number. Then $l + \epsilon$ is greater than l and thus, for some n_0 , we must have $\sup_{k \geq n_0} \{a_k\} < l + \epsilon$, since the limit of these supremums as n tends to infinity is l . Thus, for all $n \geq n_0$ we have $a_n \leq \sup_{k \geq n_0} \{a_k\} < l + \epsilon$. Similarly we have an N such that if $n > N$, then $l - \epsilon < \sup_{k \geq n} \{a_k\}$. Thus, for every $n > N$ there is a $k > n$ such that $l - \epsilon < a_k$.

Conversely suppose we know that both of the stated conditions involving ϵ are true. Then the first condition implies $\limsup_{n \rightarrow \infty} a_n \leq l + \epsilon$ for every ϵ greater than zero. We conclude from this that $\limsup_{n \rightarrow \infty} a_n \leq l$. To see that the $\limsup_{n \rightarrow \infty} a_n$ cannot be less than l , suppose it is. Then there is a positive number ϵ such that $\limsup_{n \rightarrow \infty} a_n \leq l - \epsilon$. However, this means that for n sufficiently large that $\sup_{k \geq n} \{a_k\} < l - \frac{\epsilon}{2}$. This contradicts the second assumption. Thus, we must have $\limsup_{n \rightarrow \infty} a_n = l$.

9. (15) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Show that

$$\liminf_{n \rightarrow \infty} (x_n + y_n) \geq \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n.$$

Assume that the right hand side of the inequality is never in the form $\infty + (-\infty)$.

Ans: For each $k > n$ we have $\inf_{k \geq n} \{x_k\} + \inf_{k \geq n} \{y_k\} \leq x_k + y_k$. Thus, we have

$$\inf_{k \geq n} \{x_k\} + \inf_{k \geq n} \{y_k\} \leq \inf_{k \geq n} \{x_k + y_k\}.$$

Taking the limit as $n \rightarrow \infty$ of this inequality we have

$$\lim_{n \rightarrow \infty} \inf_{k \geq n} \{x_k\} + \lim_{n \rightarrow \infty} \inf_{k \geq n} \{y_k\} \leq \lim_{n \rightarrow \infty} (\inf_{k \geq n} \{x_k + y_k\}).$$

Thus, we have shown that

$$\liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n \leq \liminf_{n \rightarrow \infty} (x_n + y_n).$$

10. (15) Show that if a Cauchy sequence has a convergent subsequence, then the original Cauchy sequence must also converge to the same limit as its convergent subsequence.

Ans: Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Suppose that $\{b_{n_k}\}_{k=1}^{\infty}$ is a subsequence of the original sequence and that $\lim_{k \rightarrow \infty} b_{n_k} = l$. Let ϵ be any positive number. Since the subsequence converges and the original sequence is Cauchy, there is a number N such that for n, m and k greater than N we have

$$|a_n - a_m| < \frac{\epsilon}{2} \quad \text{and} \quad |b_{n_k} - l| < \frac{\epsilon}{2}.$$

Thus, we have for $n > N$

$$\begin{aligned} |a_n - l| &= |a_n - b_{n_n} + b_{n_n} - l| \\ &\leq |a_n - b_{n_n}| + |b_{n_n} - l| \\ &= |a_n - a_m| + |b_{n_n} - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

where $b_{n_n} = a_m$ for some $m \geq n$.
