Each problem is worth 20 points. You must work the first two problems, and then any three of problems 3 through 7.

1. Define the following terms and give an example of each. **No** example, **no** credit.

(a) Uniformly continuous.

Ans: A function f is uniformly continuous on a set E if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x,\ y \in E$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. The function $f(x) = x$ is uniformly continuous on the entire real line.

(b) Sequentially compact.

Ans: A set E is sequentially compact if every sequence of points which belongs to the set E has a subsequence which converges to a point in E. The set which consists of the single point $x = 1$ is sequentially compact.

(c) Cluster point.

Ans: A point x_0 is a cluster point of a set E if for every $\delta > 0$, the open interval $(x_0 - \delta, x_0 + \delta)$ contains an infinite number of points of E. If $E = \{1/n : n \in \mathbb{N}\}$, then 0 is a cluster point of E.

(d) $\lim\limits_{\substack{x \to x_0 \\ x \in E}} f(x) = L$.

Ans: x_0 is assumed to be a cluster point of the set E. If both x_0 and L are finite the limit is defined as $\lim\limits_{\substack{x \to x_0 \\ x \in E}} f(x) = L$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in E$ and $0 < |x - x_0| < \delta$, then $|f(x) - L| < \epsilon$.

If x_0 is finite and L is infinite we define the limit as follows: For every M there is a δ such that if $x \in E$ and $0 < |x - x_0| < \delta$, then $f(x) > M$.

If both x_0 and E are infinite we define the limit as follows. For every M there is an N such that if $x \in E$ and $x > N$, then $f(x) > M$.

Definitions for other possibilities of x_0 and L are similar.

---

2. Prove any two of the following theorems. Be sure to say which two you want graded.

(a) If $f(x)$ is continuous on a closed bounded interval $[a, b]$, then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = sup\{ f(x) : a \leq x \leq b \}$.

Ans: Assume we have shown that if f is continuous on a closed bounded interval, then the range of f is also bounded. Let $M = sup\{ f(x) : a \leq x \leq b \}$. Then there is a sequence of points $\{x_n\}_{n=1}^{\infty} \subseteq [a, b]$, such that $\lim_{n\to\infty} f(x_n) = M$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded it possesses a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$, and this subsequence converges to some point $x_\infty \in [a, b]$. The continuity of f on the interval implies that $M = \lim_{i\to\infty} f(x_{n_i}) = f(x_\infty)$.

(b) Show that x_0 is a cluster point of a set E if and only if there is a sequence of distinct points $\{a_n\}_{n=1}^{\infty}$ of E such that $\lim_{n\to\infty} a_n = x_0$.

Ans: Suppose that x_0 is a cluster point of E. Then we can inductively construct a sequence of distinct points which converges to E. Since x_0 is a cluster point of E there is a point $a_1 \in E$ such that $|a_1 - x_0| < 1$ and $a_1 \neq x_0$. Remember every open interval about x_0 must contain an infinite number of points of E. Assume we have picked the first n points of our sequence and they satisfy $|a_i - x_0| < 1/i$, they are distinct from one-another, and none of them equals x_0. Let $\delta = \min\{ \frac{1}{n+1}, |a_i - x_0| \text{ for } 1 \leq i \leq n \}$. Then $\delta > 0$. Pick a_{n+1} from E such that $a_{n+1} \neq x_0$ and $|a_{n+1} - x_0| < \delta \leq \frac{1}{n+1}$. Thus, we have shown there is a sequence of distinct points of E which converges to the cluster point x_0.

Suppose next that x_0 is a point which is the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ of distinct points of E. Let $\delta > 0$. Then there is an N such that if $n > N$, then $|a_n - x_0| < \delta$. Thus, the open interval $(x_0 - \delta, x_0 + \delta)$ contains all of the points x_n for $n > N$. Hence this interval contains an infinite number of points of E. Remember the sequence $\{a_n\}_{n=1}^{\infty}$ consists of distinct points.

(c) Let $f(x)$ be an increasing function on the open interval (a, b).
Show that $\lim_{x \to x_0^+} f(x) = \inf\{f(x) : x_0 < x\}$ for any $x_0 \in (a, b)$.

Ans: Let $L = \inf\{f(x) : x_0 < x\}$. To see that $L = \lim_{x \to x_0^+} f(x)$, let $\epsilon > 0$. Then there exists an $x_1 > x_0$ such that $f(x_1) < L + \epsilon$. Set $\delta = x_1 - x_0$. Then if $x_0 < x < x_0 + \delta$, i.e., $x_0 < x < x_1$, we have $L \le f(x) \le f(x_1) < L + \epsilon$. Which means that $\lim_{x \to x_0^+} f(x) = L$.

Remember: you need work only three of the remaining problems.

3. Let $f(x)$ be differentiable on $(0, \infty)$ and suppose that $L = \lim_{x \to \infty} f'(x)$ exists and is finite. Prove that if $\lim_{x \to \infty} f(x)$ exists and is finite, then $L = 0$.

Ans: The fact that L must equal zero under these conditions follows from the Mean Value Theorem. For each $n \in \mathbb{N}$ we have

$$f(n+1) - f(n) = f'(c_n)\,,$$

where c_n lies between n and $n + 1$. The limit of the left hand side as $x \to \infty$ is 0. Thus, we have a sequence of points c_n which tends to infinity and for which $\lim_{n \to \infty} f'(c_n) = 0$. However, we know that the limit of $L = \lim_{x \to \infty} f'(x)$ exists. Thus, this limit must equal the limiting value of $f'(c_n) = 0$.

---

4. Evaluate the following limits.

(a) $\lim_{x \to 1} \dfrac{\ln x}{\sin(\pi x)}$.

Ans: $\lim_{x \to 1} \dfrac{\ln x}{\sin(\pi x)} = \lim_{x \to 1} \dfrac{1/x}{\pi \cos(\pi x)} = \dfrac{-1}{\pi}$.

(b) $\lim_{x \to 0^+} \dfrac{\cos x - e^x}{\ln(1 + x^2)}$.

Ans: $\lim_{x \to 0^+} \dfrac{\cos x - e^x}{\ln(1 + x^2)} = \lim_{x \to 0^+} \dfrac{-\sin x - e^x}{\frac{2x}{1+x^2}} lim_{x \to 0^+} \dfrac{-(1 + x^2)(\sin x + e^x)}{2x} = -\infty$.

(c) $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{1/x^2}$.

Ans: $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{1/x^2} = \lim_{x\to 0} e^{\frac{\ln(x/\sin x)}{x^2}}$. Taking the limit of the exponent as $x \to \infty$, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(x/\sin x)}{x^2} &= \lim_{x\to 0}\frac{\ln x - \ln\sin x}{x^2} = \lim_{x\to 0}\frac{1/x - \cos x/\sin x}{2x} \\
&= \lim_{x\to 0}\frac{\sin x - x\cos x}{2x^2\sin x} = \lim_{x\to 0}\frac{x\sin x}{4x\sin x + 2x^2\cos x} \\
&= \lim_{x\to 0}\frac{\sin x + x\cos x}{4\sin x + 8x\cos x - 2x^2\sin x} = \lim_{x\to 0}\frac{2\cos x - x\sin x}{12\cos x - 12x\sin x - 2x^2\cos x} \\
&= \frac{1}{6}\,.
\end{aligned}$$

Thus, $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{1/x^2} = e^{1/6}$.

---

5. A set of real numbers is said to be closed if it contains all of its cluster points. Show that a set which is both closed and bounded is sequentially compact.

Ans: Let E be any set which is closed and bounded. Suppose that $\{a_n\}_{n=1}^{\infty}$ is any sequence of points in E. If the sequence contains only a finite collection of points of E, then it has a subsequence which consists of only one point. This subsequence converges to this point which belongs to E. If the original sequence contains an infinite number of distinct points of E, let $\{b_n\}_{n=1}^{\infty}$ be a subsequence all of whose points are distinct. Then, since E is bounded this subsequence is bounded. By the Bolzano-Weierstrass theorem it has a convergent subsequence. Since this subsequence consists of distinct points, the value it converges to is a cluster point of E. Since E is closed this number belongs to E. Thus, the original sequence contains a subsequence which converges to a point of E. Hence E is sequentially compact.

---

6. Suppose that $f(x)$ is uniformly continuous on the open interval $(0, 1)$.

   (a) Let $\{a_n\}_{n=1}^{\infty} \subset (0, 1)$, be a Cauchy sequence. Show that $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

   Ans: To see that the sequence $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, let $\epsilon > 0$. Then there is a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. For this δ there is an N such that if $n,\ m > N$ then $|a_n - a_m| < \delta$. Thus, for this N we have $|f(a_n) - f(a_m)| < \epsilon$.

   (b) Show that $\lim_{x \to 0^+} f(x)$ exists and is finite.

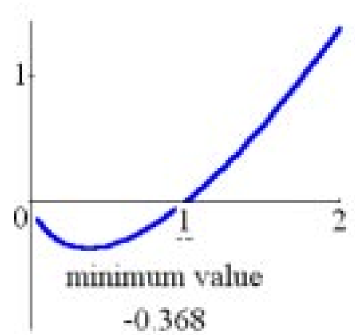
   Ans: Let $\{x_n\}$ be any sequence in $(0, 1)$ which converges to 0. This sequence is a Cauchy sequence and by the previous problem the sequence $\{f(x_n)\}$ is also Cauchy. Since every Cauchy sequence converges, this last sequence has a limit. Call it y_0 and define $f(0) = y_0$. Note that y_0 must be a finite number. We need to show that $\lim_{x \to 0^+} f(x) = y_0$. It will suffice to show that for any sequence $\{a_n\} \subset (0, 1)$ which converges to 0, that $f(a_n)$ converges to y_0. So let $\{a_n\}$ be such a sequence. Form the new sequence $c_n = x_n$ if n is odd and $c_n = a_n$ if n is even. Then c_n converges to zero and is Cauchy. Thus, $f(c_n)$ is also a Cauchy sequence and converges to some value z. However, the subsequence $f(x_n)$ converges to y_0. Thus, $z = y_0$. Moreover, the subsequence $f(a_n)$ is also Cauchy and must converge. So it too must converge to $z = y_0$. Thus, for every sequence a_n in $(0, 1)$ which converges to 0, $f(a_n)$ converges to y_0.

   ---

7. Let $f(x) = x \ln x$ for $0 < x$.

(a) Graph this function, be sure to explain why your plot looks like it does.

Ans: Before plotting this function, we gather some facts about it. First we need to decide what its limiting behavior is at $x = 0$. Thus, $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0$. Thus, the function remains bounded on every bounded interval of positive real numbers. Its limit as $x \to \infty$ is of course infinite. To find out where this function has extreme values, if any, we compute its derivative. $\frac{d}{dx}(x \ln x) = \ln x + 1$. Since $\ln x$ is a strictly increasing function on $(0, \infty)$ and takes on every real number exactly once, the derivative is negative for $x < e^{-1}$, since the derivative is zero at this value. For $x > e^{-1}$ the derivative is positive. Thus, $x \ln x$ decreases from 0 at $x = 0$ to $-e^{-1}$ at $x = e^{-1}$ and it then increases for all larger values of x. A plot of the function is shown below.



(b) Determine all intervals on which this function is monotone.

Ans: From the above discussion of the derivative of $x \ln x$ we see that $x \ln x$ is decreasing on the interval $(0, e^{-1})$ and increasing on the interval (e^{-1}, ∞).
