

Each of the first five questions is worth 20 points, and each of the last three is worth 25 points. Work the first 5 problems, and only one of the last three problems.

1. Define and give an example of each of the following. No example no credit

(a) Completeness property of the real numbers.

Ans: The completeness property says that any set E of real numbers which is bounded above has a least upper bound which we call the supremum of E .

An example which shows that the rational numbers do not have the completeness property is $E = \{x \in \mathbb{Q} : x^2 < 2\}$. E is a subset of the rational numbers and it has a supremum which is a real number that is not a rational number.

(b) $f(x)$ is Riemann integrable on the closed bounded interval $[a, b]$.

Ans: f is Riemann integrable on the interval $[a, b]$, if f is bounded on the interval, and if for every $\epsilon > 0$ there is a partition, P_ϵ , of the interval such that if Q is any refinement of P_ϵ then

$$U(f, Q) - L(f, Q) < \epsilon,$$

where $U(f, Q)$ and $L(f, Q)$ denote the upper and lower Riemann sums of f with respect to the partition Q respectively. An example of a Riemann integrable function is $f(x) = x$. This function is Riemann integrable on any bounded closed interval.

(c) The sequence of functions $f_n(x)$ converges uniformly to the function $f(x)$ on a set E .

Ans: The sequence converges uniformly if for every $\epsilon > 0$ there is an N such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for every $x \in E$. The distinction between uniform and pointwise convergence is the number N can be chosen independent of x in uniform convergence, but not in pointwise convergence. The sequence $f_n = x/n$ converges uniformly to zero on every bounded interval.

(d) The infinite series $\sum_{n=1}^{\infty} a_n$ converges to the finite real number l .

Ans: Convergence of an infinite series is defined in terms of the convergence of the sequence of its partial sums. That is, let $S_n = \sum_{k=1}^n a_k$, then

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} S_n.$$

An example of a convergent series is $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

2. Let $f(x)$ be differentiable on the interval $[a, b]$. Show that $f'(x) \geq 0$ on the interval $[a, b]$ if and only if $f(x)$ is monotone increasing on the interval $[a, b]$.

Ans: Suppose first that f is monotone increasing on the interval. Then for $h > 0$ we have

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \geq 0,$$

since both the numerator and denominator of the quotient are nonnegative. Conversely if we know that $f' \geq 0$, then we have for $x_1 > x_2$, $f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2) \geq 0$. Thus, f is monotone increasing if its derivative is nonnegative.

-
3. Let $a_n = \frac{4 - 2n}{n + 1}$. Find the limit as n goes to infinity of the a_n , and then use the definition of limit to show that your answer is correct.

Ans: The limiting value of the sequence is -2 . Verifying that the condition in the definition of a limit is met, let $\epsilon > 0$. Let $N > \frac{3}{\epsilon}$. Then for $n > N$ we have

$$\left| \frac{4 - 2n}{n + 1} + 2 \right| = \left| \frac{3}{n + 1} \right| < \epsilon.$$

-
4. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{(\pi)^k}$ converges, and find its value.

Ans: The series is a geometric series with $r = \frac{-1}{\pi} < 1$. Thus, it converges, and its value is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{(\pi)^k} &= - \sum_{n=1}^{\infty} \left(\frac{-1}{\pi} \right)^k \\ &= - \frac{-1/\pi}{1 - (-1/\pi)} \\ &= \frac{1}{\pi + 1}. \end{aligned}$$

5. Let f and g be Riemann integrable on the interval $[a, b]$, and suppose that $f(x) \leq g(x)$ for all $x \in [a, b]$. Using the definition of $\int_a^b f(x) dx$, show that

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Ans: Since both f and g are Riemann integrable the value of their integrals equals the infimum of their upper Riemann sums. Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of the interval $[a, b]$. Let $M_j(f)$ denote the supremum of f on the j^{th} subinterval of $[a, b]$ determined by the partition P . Then for each j we have $M_j(f) \leq M_j(g)$. This implies the following inequalities:

$$\int_a^b f(x) dx \leq \sum_{j=1}^n M_j(f) \leq \sum_{j=1}^n M_j(g).$$

Taking the infimum of the right hand side we have the inequality

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Remember, work only one of the following problems.

6. Determine whether or not each of the following series converges absolutely, conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n}$.

Ans: This series diverges. Since $\frac{n^{1/n}}{n} \geq \frac{1}{n}$ for each n , and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, this series must also diverge by the comparison test.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$.

Ans: This series converges absolutely. One way to see this is to use the ratio test.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^2/2^{n+1}}{n^2/2^n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{n+1}{n} \right)^2 = \frac{1}{2}. \end{aligned}$$

Since the limiting value is less than 1, the series converges absolutely.

7. If $f(x)$ is integrable on the interval $[0, 1]$, show that

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) dx = 0.$$

Ans: Since f is integrable it is bounded. Let M be such that $|f(x)| \leq M$ for all x in the interval $[0, 1]$. Let $\epsilon > 0$. Pick $\delta > 0$ such that $\delta < \frac{\epsilon/2}{M}$. The functions x^n converge to zero uniformly on the interval $[0, 1 - \delta]$. Thus, we can pick N such that if $n > N$ then $|x^n| < \frac{\epsilon/2}{M(1-\delta)}$. For $n > N$, we have

$$\begin{aligned} \left| \int_0^1 x^n f(x) dx \right| &= \left| \int_0^{1-\delta} x^n f(x) dx + \int_{1-\delta}^1 x^n f(x) dx \right| \\ &\leq \int_0^{1-\delta} |x^n f(x)| dx + \int_{1-\delta}^1 |x^n f(x)| dx \\ &\leq \int_0^{1-\delta} \frac{\epsilon/2}{1-\delta} dx + \int_{1-\delta}^1 M dx \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

8. Suppose that $f_n(x)$ converges uniformly to the function $f(x)$ on the interval $[a, b]$, and suppose in addition that each of the functions f_n is continuous on this interval. Show that the limit function f must also be continuous.

Ans: Let x_0 be any point in $[a, b]$. Let $\epsilon > 0$. Since the sequence of functions f_n converges to f uniformly, there is an n such that $|f_n(x) - f(x)| < \epsilon/3$ for all x in the interval $[a, b]$. Moreover since f_n is continuous at the point x_0 there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f_n(x) - f_n(x_0)| < \epsilon/3$. Thus, we have for $|x - x_0| < \delta$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$
