1. (20) Define or state the following:

   (a) $\lim_{n \to \infty} a_n = l$, Note, there are 3 cases here, $l = \pm\infty$ or l finite, define the limit for each of these cases.

   l finite For all $\epsilon > 0$, $\exists N$ such that if $n > N$, then $|a_n - l| < \epsilon$.

   l postive infinity For all M there is an N such that if $n > N$, than $a_n > M$.

   l negative infinity For all M there is an N such that if $n > N$, than $a_n < M$.

   (b) Cauchy sequence,

   A sequence a_n is Cauchy if $\forall \epsilon > 0$, $\exists N$ such that if n, $m > N$ then

$$|a_n - a_m| < \epsilon \ .$$

   (c) Bolzano-Weierstrass theorem,

   Every bounded sequence of real numbers contains a convergent subsequence.

   (d) the function f is continuous at the point a.

   For every $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

2. (10) Suppose that $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$. Show that $\lim_{n \to \infty} (a_n + b_n) = A + B$.

   Let $\epsilon > 0$. Then there exist N_1 and N_2 such that if $n > N_1$ or $n > N_2$ then

$$|a_n - A| < \frac{\epsilon}{2} \text{ or } |b_n - B| < \frac{\epsilon}{2},$$

respectively. Then for $n > \max(N_1, N_2)$ we have

$$\begin{aligned} |(a_n + b_n) - (A + B)| &\leq |a_n - A| + |b_n - B| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

3. (20) Let E be a bounded set of real numbers.

   (a) Define the infimum of the set E.

   A real number l is the infimum of a set E if l is a lower bound of E, that is $l \leq x$ for all $x \in E$, and if k is any other lower bound of E, then $k \leq l$.

   (b) Show that there is a sequence of points, a_n, in E such that

$$\lim_{n \to \infty} a_n = \inf(E) \ .$$

   Set $s = \inf(E)$. Since E is bounded below s is a real number and for any $\epsilon > 0$ there is an x in E such that $s \leq x < s + \epsilon$. For otherwise $s + \epsilon$ is a lower bound of E strictly greater than the greatest lower bound or infimum of E. Thus, for each $n \in N$ there exists an $a_n \in E$ such that $s \leq a_n < s + 1/n$. By the squeeze theorem we have $\lim_{n \to \infty} a_n = s$.

4. (10) State and prove the monotone convergence theorem for decreasing sequences of real numbers.

   If a_n is a monotone decreasing sequence of real numbers that is bounded below, then the sequence converges. To prove this theorem let $\epsilon > 0$. Then there is an N such that $\inf\{a_n\} \leq a_N < \inf\{a_n\} + \epsilon$. Since the sequence a_n is decreasing we have for all $m \geq N$ that $\inf\{a_n\} \leq a_m < \inf\{a_n\} + \epsilon$. Thus, for all $n \geq N$ we have

$$|a_n - \inf\{a_n\}| < \epsilon \ .$$

   Thus, $\lim_{n \to \infty} a_n = \inf\{a_n\}$.

5. (15) Set $x_1 = 1$, and $x_{n+1} = \sqrt{2 + x_n}$ for $n = 1, 2, \cdots$.

(a) Show that x_n converges to some number l.

To see that this sequence converges we'll show that it is an increasing sequence bounded above by 2. The first step is to show that $1 \le x_n \le 2$ for all n. It is true for $n = 1$, so assume it's true for n. Then we have

$$1 \le \sqrt{2 + x_n} \le \sqrt{2 + 2} = 2 \; .$$

Since $x_{n+1} = \sqrt{2 + x_n}$, we have an induction proof that $1 \le x_n \le 2$ is true for all $n \in N$. To see that this sequence is increasing we note

$$x_{n+1}^2 = 2 + x_n \ge x_n + x_n = 2x_n \ge x_n^2 \; .$$

Since $x_{n+1} = \sqrt{x_{n+1}^2} \ge \sqrt{x_n^2} = x_n$, we conclude that the sequence is increasing. Thus, by the monotone convergence theorem we know that the sequence converges.

(b) Determine the value of l.

Taking the limit of the recursive equation we have

$$l = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \sqrt{2 + x_n} = \sqrt{2 + l}.$$

This implies that $l^2 = 2 + l$, which in turn implies that $l = -1$ or 2. Since $l > 0$, we must have $l = 2$.

6. (15) Use the definition of limit to show that

$$\lim_{n \to \infty} \frac{2n + 1}{3 + n} = 2 \; .$$

Let $N = \dfrac{5}{\epsilon}$. If $n > N$ we have

$$\begin{aligned}
\left| \frac{2n+1}{3+n} - 2 \right| &= \left| \frac{2n + 1 - 6 - 2n}{3+n} \right| = \left| \frac{-5}{3+n} \right| \\
&= \frac{5}{3+n} < \frac{5}{n} < \frac{5}{N} = \frac{5}{(5/\epsilon)} = \epsilon \; .
\end{aligned}$$

7. (10) Let $f(x) = \begin{cases} 2x-1, & x<2 \\ 3, & x=2 \\ x^2, & 2<x \end{cases}$.

(a) Explain why f is continuous at every point x, except $x = 2$.

At every point except 2, the function f is like a polynomial in x. For $x < 2$ it's $2x - 1$, and for $x > 2$ it's x^2. Such functions are continuous everywhere so f must be continuous at all $x \neq 2$.

(b) What are the left and right hand limits at $x = 2$. In addition to determining these limits, use the definition to prove one of your answers.

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= 3 \\ \lim_{x \to 2^+} f(x) &= 4\,. \end{aligned}$$

The following is a verification that $\lim_{x \to 2^-} f(x) = 3$. Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{2}$. Then if $2 - \delta < x < 2$ we have

$$\begin{aligned} |f(x) - 3| &= |(2x-1) - 3| \\ &= |2x - 4| \\ &= 2|x-2| < 2\frac{\epsilon}{2} = \epsilon. \end{aligned}$$