1. (15) State and prove Rolle's theorem

   Rolle's theorem: Let f be continuous on the closed bounded interval $[a,\, b]$ and differentiable on the open interval $(a,\, b)$. If $f\,(a) = f\,(b)$, then there is a point $\xi \in (a, b)$ such that $f\,'\,(\xi) = 0$. Note: it is implicitly assumed that $a < b$.

   To prove Rolle's theorem assume first that either the global maximum or global minimum value of f occurs at an interior point ξ. Then this point ξ is the location of a local extremum, and we must have $f\,'\,(\xi) = 0$. If neither the global max or min occurs at an interior point, then they occur at a or b. Since $f\,(a) = f\,(b)$ this means that the maximum and minimum values of f are the same, which implies that f is a constant. Hence its derivative is identically zero, and any point $\xi \in (a,\, b)$ works.

   The fact that f must attain its supremum and infimum follows from the fact that f is continuous on the closed bounded interval $[a,\, b]$.

2. (15) Suppose f is differentiable at the point x_0 and has a local maximum there.

   (a) Define what local maximum means,

   f has a local maximum at x_0 means that there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $f\,(x) \le f\,(x_0)$.

   (b) Show $f\,'\,(x_0) = 0$.

   Since f is differentiable at x_0 the limit of the difference quotient both from above and below exists, and these two values must be equal. Since we have a local maximum at x_0, for x close to x_0 we have $f\,(x) - f\,(x_0) \le 0$. Thus,

$$\begin{aligned} f\,'\,(x_0) &= \lim_{x \,\to\, x_0^+} \frac{f\,(x) - f\,(x_0)}{x - x_0} \le 0 \text{ and} \\ f\,'\,(x_0) &= \lim_{x \,\to\, x_0^-} \frac{f\,(x) - f\,(x_0)}{x - x_0} \ge 0 \;, \end{aligned}$$

   and the derivative must equal 0.

3. (10) Use the definition of the derivative to prove the following formula. Be sure to state any conditions needed for the formula to be true

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx} \ .$$

Suppose that both f and g are differentiable at the point x_0, then the product fg is also differentiable at this point. Moreover

$$\begin{aligned}
\left.\frac{d}{dx}(fg)\right|_{x_0} &= \lim_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x) + f(x_0)\,g(x) - f(x_0)\,g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} g(x) + \lim_{x \to x_0} f(x_0) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= \left.\frac{df}{dx}\right|_{x_0} g(x_0) + f(x_0) \left.\frac{dg}{dx}\right|_{x_0} \ .
\end{aligned}$$

The fact that $\lim_{x \to x_0} g(x) = g(x_0)$ follows from the fact that if a function is differentiable at a point, then it is continuous at this point.

4. (15) Suppose you know the function f is continuous and differentiable on the interval (a, b), and that its derivative has the following properties:

on (a, c) the derivative is negative and on (c, b) the derivative is positive,

for some point c between a and b. Show the following:

(a) for all $x \in (a, b)$ we have $f(x) \geq f(c)$,

Using the mean value theorem we have

$$f(x) - f(c) = f'(\xi)(x - c) \ .$$

Regardless of whether $x \leq c$ or $x \geq c$ the expression $f'(\xi)(x - c)$ is non-negative. Thus, we must have $f(x) - f(c) \geq 0$.

(b) $f'(c) = 0$,

From part a. we see that f has a local minimum at $x = c$. Hence its derivative must be zero at $x = c$.

(c) show that for all $x \in R$ we have $\sin^2(x) \le 2|x|$.

Setting $f(x) = 2|x| - \sin^2(x)$, we want to show that this function is always non-negative. Since f is an even function we only need to show that it is non-negative for $x \ge 0$. If $x \ge 0$ we have

$$\begin{aligned} f(x) &= f(x) - f(0) = f'(\xi)\, x \\ &= (2 - 2\sin\xi\cos\xi)\, x \\ &\ge 0\ . \end{aligned}$$

Since the absolute value of sin and cos is always less than 1, we must have $(2 - 2\sin\xi\cos\xi) \ge 0$. From $f \ge 0$ we have

$$\sin^2 x \le 2|x|\ .$$

5. (15) Let $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$. For which x is f differentiable, and at which x is the derivative continuous ?

It turns out that f is differentiable everywhere and its derivative is continous everywhere except at $x = 0$. First we calculate the derivative of f at $x = 0$.

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin\dfrac{1}{x}}{x} = \lim_{x \to 0} \left(x \sin\frac{1}{x} \right) = 0\ .$$

Thus, the derivative of f equals

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Since $2x\sin(1/x) - \cos(1/x)$ is a continuous function of x for $x \ne 0$, and the limit of this function as x approaches 0 does not exist, we see that the derivative of f, while existing everywhere, is not continuous at the origin.

6. (20) Evaluate, if possible, the following limits:

(a) $\displaystyle\lim_{x \to 0} \frac{1 - \cos(3x)}{x^2}$

$$\begin{aligned} \lim_{x \to 0} \frac{1 - \cos(3x)}{x^2} &= \lim_{x \to 0} \frac{3\sin(3x)}{2x} = \lim_{x \to 0} \frac{9\cos(3x)}{2} \\ &= \frac{9}{2}\ . \end{aligned}$$

(b) $\displaystyle\lim_{x \to -1} \frac{\ln|x|}{x+1}$

$$\begin{aligned} \lim_{x \to -1} \frac{\ln|x|}{x+1} &= \lim_{x \to -1} \frac{\ln(-x)}{x+1} = \lim_{x \to -1} \frac{\dfrac{-1}{-x}}{1} \\ &= \lim_{x \to -1} \frac{1}{x} = -1\ . \end{aligned}$$

7. (10) Determine whether the following statements are true or false. If true supply a proof, and if false a counterexample.

   (a) If $f(x)$ is continuous on the interval $[a, b]$, then it has a derivative at each $x \in (a, b)$.

   This is not true. $f(x) = |x|$ is continuous everywhere, but not differentiable at $x = 0$.

   (b) Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and there is a point $\xi \in (a, b)$ such that $f'(\xi) = 0$. Then the mean value theorem tells us that we must have $f(a) = f(b)$.

   This is also false. Set $f(x) = x^2$ on the interval $[-1, 2]$. Then $f(-1) = 1 \neq 4 = f(2)$, but $f'(0) = 0$.