1. (25) State and prove the monotone convergence theorem. Then use it to show that the recursive sequence defined below converges to some number l. What must l equal?

$$a_1 = 1,\ a_{n+1} = \sqrt{2 + a_n}\ .$$

The monotone convergence theorem says the following: let a_n be a monotone sequence of numbers. That is, either $a_n \le a_{n+1}$ or $a_n \ge a_{n+1}$ for each n. If the sequence is bounded, then it converges.

For convenience, let's assume that the sequence is monotone increasing and bounded above. By the completeness axiom of real numbers, the collection of numbers $\{a_n\}_{n=1}^{\infty}$ has a least upper bound. Call it l. Note, since the numbers a_n are bounded above we have l a finite real number. To see that a_n converges to l, first note that $a_n \le l$ for each n. For if there is some N such that $l < a_N$, then l is not an upper bound of the set $\{a_n\}_{n=1}^{\infty}$. Secondly, for any $\epsilon > 0$ there is an N such that $l - \epsilon < a_N$. For if not, then $l - \epsilon$ is an upper bound of the set $\{a_n\}_{n=1}^{\infty}$, contradicting the fact that l is the least upper bound for this set. Now let $\epsilon > 0$. Let N be such that

$$l - \epsilon < a_N.$$

Then for all $n \ge N$ we have

$$l - \epsilon < a_N \le a_n \le l < l + \epsilon\ .$$

Thus, for $n \ge N$ we have $|a_n - l| < \epsilon$. That is, the sequence a_n converges to l.

To see that the sequence $a_1 = 1$, and $a_{n+1} = \sqrt{2 + a_n}$ converges, we'll show that it is monotone increasing and bounded above by 2. To see that 2 is an upper bound for the numbers a_n, note that $a_1 = 1 \le 2$. Now suppose that $a_n \le 2$. Then we have

$$a_{n+1} = \sqrt{a_n + 2} \le \sqrt{2 + 2} = 2.$$

So by induction we have $a_n \le 2$ for all n. To see that a_n is increasing note that $a_2 = \sqrt{3} \ge 1 = a_1$. Using induction again suppose that $a_k \ge a_{k-1}$. Then

$$a_{k+1} = \sqrt{2 + a_k} \ge \sqrt{2 + a_{k-1}} = a_k.$$

Since the sequence is bounded and increasing, the monotone convergence theorem implies that the sequence has a limit. Call it l. Then l must satisfy

$$\begin{aligned} l &= \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 2} = \sqrt{l + 2} \\ l^2 &= l + 2. \end{aligned}$$

The solutions to this quadratic equation are 2 and -1. Since $a_n > 0$ for all n, we must have $l = 2$.

2. (25) State and prove Rolle's theorem. Next, suppose a function f has two derivatives on an interval $[a,\, b]$, and there are three points x_i, such that $a < x_1 < x_2 < x_3 < b$ for which $f(x_1) = f(x_2) = f(x_3)$. Show such a function must have a point ξ at which $f''(\xi) = 0$, where $x_1 < \xi < x_3$.

Rolle's theorem states that if f is continuous on the closed bounded interval $[a,\, b]$ and differentiable on the open interval $(a,\, b)$, and $f(a) = f(b)$, then there is a point $\xi \in (a,\, b)$ such that $f'(\xi) = 0$.

One proof of Rolle's theorem goes as follows. Since f is continuous on the closed bounded interval there are points x_m and x_M in $[a,\, b]$, where f attains its infimum and supremum respectively. If one of those points is in the open interval, then f has a local extremum at that point, in which case f 's derivative is zero there. On the other hand if neither of the points x_m or x_M is in the open interval $(a,\, b)$, then both of them are one of the endpoints. However, since $f(a) = f(b)$ this implies that we have

$$f(x_m) \le f(x) \le f(x_M) = f(x_m) \ .$$

That is, f is constant on the closed interval $[a,\, b]$. In this case we have $f'(x) = 0$ for all $x \in (a,\, b)$, and we may pick ξ equal to any interior point.

For the function with the properties described above we have $f(x_1) = f(x_2) = f(x_3)$. Thus, by Rolle's theorem there are points ξ_1 and ξ_2 such that

$$x_1 < \xi_1 < x_2 < \xi_2 < x_3,$$

and $f'(\xi_1) = 0 = f'(\xi_2)$. A second application of Rolle's theorem to the function $f'(x)$ says there is a point ξ, where $\xi_1 < \xi < \xi_2$ and $f''(\xi) = 0$.

3. (30) Define what it means for a bounded function f to be Riemann integrable on a bounded interval $[a,\, b]$. Using the definition, show that

$$f(x) = \begin{cases} 5, & 0 \le x < 1 \\ -1, & x = 1 \\ 3, & 1 < x \le 2 \end{cases}$$

is Riemann integrable on $[0,\, 2]$, and calculate $\int_0^2 f(x)\ dx$.

A bounded function defined on the interval $[a,\, b]$ is Riemann integrable on this interval if for all $\epsilon > 0$, there is a partition $P = \{x_0,\ \cdots,\ x_n\}$ of the interval such that

$$U(f,\, P) - L(f,\, P) < \epsilon,$$

where $U(f,\, P)$ and $L(f,\, P)$ are the upper and lower Riemann sums of f with respect to the partition P. That is,

$$\begin{aligned} U(f,\, P) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \\ L(f,\, P) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \end{aligned}$$

where $M_i(f) = \sup\{f(x) : x_{i-1} \le x \le x_i\}$ and $m_i(f) = \inf\{f(x) : x_{i-1} \le x \le x_i\}$.

To see that the given f is Riemann integrable on $[0, 1]$, let $\epsilon > 0$, and assume that $\epsilon < 14$. Let

$$P = \left\{0,\ 1 - \frac{\epsilon}{14},\ 1 + \frac{\epsilon}{14},\ 2\right\}.$$

The partition P gives us 3 subintervals of the interval $[0, 2]$. We have

$$\begin{aligned} M_1(f) &= 5,\ M_2(f) = 5,\ M_3(f) = 3 \\ m_1(f) &= 5,\ m_2(f) = -1,\ m_3(f) = 3. \end{aligned}$$

Thus,

$$\begin{aligned} U(f, P) - L(f, P) &= \left[5 * \left(1 - \frac{\epsilon}{14}\right) + 5 * \left(\frac{\epsilon}{7}\right) + 3\left(2 - \left(1 + \frac{\epsilon}{14}\right)\right)\right] \\ &\quad - \left[5 * \left(1 - \frac{\epsilon}{14}\right) + (-1) * \left(\frac{\epsilon}{7}\right) + 3\left(2 - \left(1 + \frac{\epsilon}{17}\right)\right)\right] \\ &= 6\left(\frac{\epsilon}{7}\right) < \epsilon. \end{aligned}$$

Now that we know f is Riemann integrable on $[0, 2]$, we can use some results of Riemann integration to evaluate the integral

$$\begin{aligned} \int_0^2 f(x)\ dx &= \int_0^1 f(x)\ dx + \int_1^2 f(x)\ dx \\ &= \lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} f(x)\ dx + \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^2 f(x)\ dx \\ &= \lim_{\epsilon \to 0^+} 5(1 - \epsilon) + \lim_{\epsilon \to 0^+} 3(1 - \epsilon) \\ &= 8. \end{aligned}$$

4. (30) Suppose f and g are bounded functions on $[a, b]$ and that both are Riemann integrable on $[a, b]$. Determine the the truth or falsity of each of the following statements. If a statement is true give a proof; if it is false supply a counterexample.

   (a) If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$.

   This statement is true. One way to see this is to note that since the functions are integrable, their integrals equal the limit of a sequence of Riemann sums. And for any Riemann sum we have

$$\sum_{i=1}^n f(t_i)\,\Delta x_i \le \sum_{i=1}^n g(t_i)\,\Delta x_i.$$

   Taking limits we get $\int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$.

   (b) If $f(x) \ge 0\ \forall x \in [a, b]$ and $\int_a^b f(x)\, g(x)\ dx = 0$, then $g(x) = 0\ \forall x \in [a, b]$.

   This is not true. A counter example is $f(x) = 1$ on $[-1, 1]$ and $g(x) = x$.

(c) Let $F(x) = \int_a^x f(t)\ dt$ for $x \in [a, b]$. Then $F'(x) = f(x)\ \forall x \in (a, b)$.

This is not true. If we assume that f is continuous then it is true, but if f is not continous at a point x_0, then $F'(x_0)$ may not exist. As an example consider the function f in problem 3. For that f, we have

$$F(x) = \begin{cases} 5x, & 0 \le x \le 1 \\ 3(x-1)+5, & 1 < x \le 2 \end{cases} .$$

Note that $F'(1)$ does not exist, since the right hand and left hand limits of the difference quotient are not equal

$$\begin{aligned} \lim_{h \to 0^+} \frac{F(1+h) - F(1)}{h} &= \lim_{h \to 0^+} \frac{3h}{h} = 3 \\ \lim_{h \to 0^-} \frac{F(1+h) - F(1)}{h} &= \lim_{h \to 0^-} \frac{5h}{h} = 5 . \end{aligned}$$

5. (20) Suppose that f is continuous on the open interval $(0, 1)$, and there is a constant $K > 0$ such that $\forall x \in (0, 1)$, $|f(x)| \le K$. For each $x \in (0, 1)$, define $F(x) = \int_{1/2}^x f(t)\ dt$.

(a) Show that there is a constant $\hat{K}$ such that for any x and y in $(0, 1)$ we have

$$|F(x) - F(y)| \le \hat{K}\,|x - y| .$$

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_{1/2}^x f(t)\ dt - \int_{1/2}^y f(t)\ dt \right| \\ &= \left| \int_y^x f(t)\ dt \right| \le \int_{\min\{x,\, y\}}^{\max\{x,\, y\}} |f(t)|\ dt \\ &\le \int_{\min\{x,\, y\}}^{\max\{x,\, y\}} K\ dt = K\,|x - y| . \end{aligned}$$

Set $\hat{K}$ equal to K.

(b) Deduce that F is uniformly continuous on the open interval $(0, 1)$, and that it has a continuous extension to the closed interval $[0, 1]$. Thus, showing that f is improperly Riemann integrable on the closed interval $[0, 1]$.

Clearly this inequality implies that F is uniformly continuous on $(0, 1)$. To see this implies that F has a continuous extension to the closed interval, we'll show that for any sequence of points $x_n \in (0, 1)$ that converges to 1 or 0, that $F(x_n)$ must also converge. Since x_n converges it is a Cauchy sequence; the fact that F is uniformly continuous, or use the inequality that F satisfies, implies that $F(x_n)$ is a Cauchy sequence, and this implies that $F(x_n)$ converges. We need to show that this limiting value of F does not depend upon the sequence x_n that converges to one of the interval endpoints. So suppose x_n and y_n both converge to 1. Consider the sequence $x_1, y_1, x_2, y_2, \cdots$. That is $z_{2n-1} = x_n$, and $z_{2n} = y_n$ for any $n \in N$. Then z_n converges to 1, and $F(z_n)$ must also converge. This means that $F(x_n)$ and $F(y_n)$ have to converge to the same value, since subsequences of a convergent sequence converge to the same limit as the original sequence.

6. (20) Define what it means to say that the sequence of functions f_n converges uniformly to f on the set E. Show that the sequence f_n, defined below, converges uniformly to $f(x) = x$, on the set $E = [0, 1]$.

$$f_n(x) = n \sin \frac{x}{n} \ .$$

A sequence of functions f_n converges uniformly to f on a set E if for every $\epsilon > 0$ there is an N such that if $n > N$, then

$$|f_n(x) - f(x)| < \epsilon \ \ \forall x \in E \ .$$

To see that the given sequence converges uniformly on the interval $[0, 1]$, set $h(x) = \sin x - x$. Then we have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| n \sin \frac{x}{n} - x \right| \\ &= n \left| \sin \frac{x}{n} - \frac{x}{n} \right| = n \left| h\left(\frac{x}{n}\right) - h(0) \right| \\ &= n \left| h'(\xi) \frac{x}{n} \right| = |\cos \xi - 1| \, |x| \\ &\leq |\cos \xi - 1| \, . \end{aligned}$$

Since $0 < \xi < \dfrac{x}{n} \leq \dfrac{1}{n}$ we see that as $n \to \infty$ the value ξ goes to 0 independent of x. Thus, we have $\cos \xi - 1$ tending to 0 uniformly in $x \in [0, 1]$.