1. (30)

a. What is the definition of a function being Riemann integrable on the interval $[a, b]$? Be sure to state any conditions, which the function *f* or the interval $[a, b]$ are supposed to satisfy.

b. Let $f \in C[a, b]$. Show that *f* is Riemann integrable on $[a, b]$.

See the text for the definition and proof.

2. (35)

a. State the fundamental theorem of calculus. There are two statements, be sure to give them both, and then prove one of the two statements.

See the text for the statements and their proofs.

b. Find a formula for $\frac{d}{dt}\int_0^t g(x-t)dx$. In proving your formula you may assume that $g \in C(R)$.

Since we're not assuming that the function *g* is differentiable we need to remove the *t* variable from the argument of *g*. To do this we use the change of variables formula

$$\int_{\phi(a)}^{\phi(b)} g(u)\,du = \int_a^b g(\phi(x))\phi'(x), dx.$$

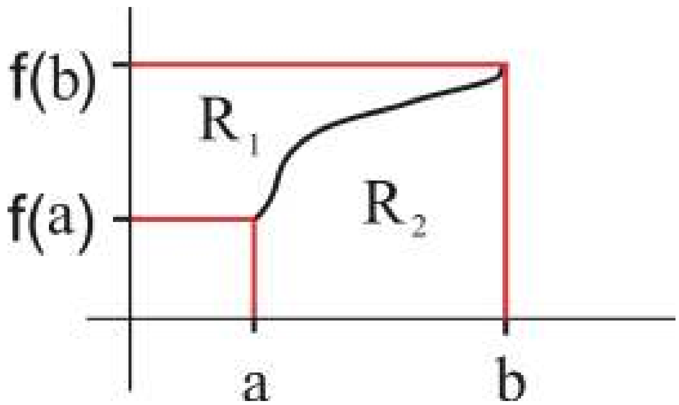
Set $\phi(x) = x - t$, then we have $\phi(-t) = 0$, $\phi(0) = t$, and $\phi'(x) = 1$.

$$\begin{aligned}\frac{d}{dt}\int_0^t g(x-t)dx &= \frac{d}{dt}\left(\int_{-t}^0 g(x)\,dx\right)\\ &= -(-g(-t))\\ &= g(-t).\end{aligned}$$

c. Suppose that f is one-to-one and continuously differentiable on $\left[a, b\right]$. Show that

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a) .$$

The best way to see why this formula is true is with a picture, which is shown below. Following the picture is a "real" proof.



Notice that the area of the region labeled R_1 is $\int_{f(a)}^{f(b)} f^{-1}(x)dx$ and the area of the region labeled R_2 is $\int_a^b f(x)dx$ and clearly the sum of these two areas equals $bf(b) - af(a)$. Now for a "real" proof.

Set $F(t) = \int_a^t f(x)dx + \int_{f(a)}^{f(t)} f^{-1}(x)dx - (tf(t) - af(a))$. Note that $F(a) = 0$. Taking the derivative of F we have

$$\frac{d}{dt}F(t) = \frac{d}{dt}\left\{\int_a^t f(x)dx + \int_{f(a)}^{f(t)} f^{-1}(x)dx - (tf(t) - af(a))\right\}$$

$$= f(t) + f^{-1}(f(t))f'(t) - (f(t) + tf'(t))$$
$$= f(t) + tf'(t) - (f(t) + tf'(t))$$
$$= 0 .$$

Since $F'(t)$ is zero, F must be a constant and from $F(0) = 0$, we have $F(t) = 0$. Thus,

$$0 = F(b) = \int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx - (bf(b) - af(a)) \text{ or}$$

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$$

3. (15) A function is said to be odd if $f(-x) = -f(x)$ for all $x \in R$, and it is said to be even if $f(-x) = f(x)$ for all $x \in R$. If f is an even function, which is differentiable everywhere, show that its derivative is an odd function.

$$\frac{d}{dx} f(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(-(x-h)) - f(-x)}{h}$$

$$= \lim_{h \to 0} \frac{f((x-h)) - f(x)}{h} = -\lim_{h \to 0} \frac{f((x+(-h))) - f(x)}{-h}$$

$$= -f'(x)$$

4. (10) Suppose that g is integrable and nonnegative on the interval $\left[1, 3\right]$ with $\int_1^3 g(x)dx = 1$. Prove that

$$\frac{1}{\pi} \int_1^9 g(\sqrt{x})dx < 2 .$$

This also calls for a change of variables, and we have

$$\frac{1}{\pi} \int_1^9 g(\sqrt{x})dx = \frac{1}{\pi} \int_1^3 g(u)(2u)du = \frac{2}{\pi} \int_1^3 ug(u)du$$

$$\leq \frac{2 \cdot 3}{\pi} \int_1^3 g(u)du = \frac{2 \cdot 3}{\pi}$$

$$< 2 .$$

5. (10) Suppose $f \in C^1(R)$ and that there is a $\delta > 0$ such that $f'(x) \geq \delta$ for all $x \in R$. Show that

$$\lim_{x \to \infty} f(x) = \infty .$$

Let x_0 be any point in R. Then we have

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\, dt \geq \int_{x_0}^{x} \delta\, dt = \delta(x - x_0) .$$

Thus, we have $f(x) \geq \delta x + f(x_0) - \delta x_0$. Since $\delta > 0$, the right hand side of this inequality goes to infinity as x goes to infinity, which forces $f(x)$ to go to infinity as x goes to infinity.