1. (20) Let $0 < a_1 \leq 3$. Define $a_{n+1} = \sqrt{2a_n + 3}$.

   (a) State the monotone convergence theorem.

   Let $\{a_n\}_{n=1}^{\infty}$ be a monotone sequence, either increasing or decreasing. If the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded then $\lim_{n\to\infty} a_n$ exists.

   (b) Prove that the sequence a_n converges and find its limiting value.

   We will show that the given sequence is increasing and bounded above by 3. Thus, its limit, which we denote by L, exists and in fact equals 3. To see that a_n is bounded above by 3 we use induction. By hypothesis a_1 is less than or equal to 3. Suppose now that $a_k \leq 3$. Then we have

$$\begin{aligned} a_{k+1}^2 &= 2a_k + 3 \leq 2\cdot 3 + 3 = 9 \text{ so} \\ a_{k+1} &\leq 3\ . \end{aligned}$$

   By induction we have that all of the a_n are less than or equal to 3. To see that the sequence is increasing we note

$$\begin{aligned} a_{k+1}^2 &= 2a_k + 3 \geq 2a_k + a_k = 3a_k \geq a_k^2\ . \text{ Thus} \\ a_{k+1} &\geq a_k\ . \end{aligned}$$

   Take the limit as $n \to \infty$ of the following

$$L^2 = \lim_{n\to\infty} a_{k+1}^2 = \lim_{n\to\infty} 2a_k + 3 = 2L + 3.$$

   Thus, $L = 3$ or $L = -1$. Since $a_1 > 0$, $L > 0$, and we can conclude that $L = 3$

2. (20) Show that $\dfrac{\sin x}{x}$ is uniformly continuous on the open interval (0, 1).

   First note that $\lim_{x\to 0^+} \dfrac{\sin x}{x} = 1$, and that $\sin x$ is continuous on $(0, \infty)$. Thus, $\sin x$ is uniformly continuous on any interval of the form $[\delta, 1]$, for any δ such that $0 < \delta < 1$. Now let $\epsilon > 0$. There exists a $\delta_1 > 0$ such that $\left|\dfrac{\sin x}{x} - 1\right| < \epsilon/2$ if $0 < x < \delta_1$. Thus, for x and y both less than this δ_1 we have

$$\left|\frac{\sin x}{x} - \frac{\sin y}{y}\right| < \epsilon$$

   Now there is a δ_2 such that for x, y in the interval $[\delta_1/2,\ 1]$, if $|x - y| < \delta_2$, then $\left|\dfrac{\sin x}{x} - \dfrac{\sin y}{y}\right| < \epsilon$. Now let $\delta = \min\{\delta_1,\ \delta_2\}$. Suppose now x and y belong to (0, 1) and $|x - y| < \delta$. If one of these numbers is in the interval $(0,\ \delta_1/2)$, then both are in the interval $(0,\ \delta_1)$ and hence we have $\left|\dfrac{\sin x}{x} - \dfrac{\sin y}{y}\right| < \epsilon$. If neither of them is in $(0,\ \delta_1/2)$, then both are in the interval $[\delta_1/2,\ 1]$ and again we have $\left|\dfrac{\sin x}{x} - \dfrac{\sin y}{y}\right| < \epsilon$. Thus, $\dfrac{\sin x}{x}$ is uniformly continuous on (0, 1).

3. (20) State and prove Rolle's theorem.

Rolle's theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and if $f(a) = f(b)$, then there is a point $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Since f is continuous on the closed interval there are values ξ_1 and ξ_2 where f takes on its maximum and minimum values. If these extreme values are equal then f is constant, and its derivative is identically zero. So every point in the open interval (a, b) is an acceptable ξ. We can, therefore, assume that the extreme values are not equal, and hence at least one of them is not equal to $f(a) = f(b)$, which means that one of the ξ_i must be in the open interval, and assume it's ξ_1. Since f is differentiable at all points of the open interval and the derivative of any function at a local extremum is zero, we have $f'(\xi_1) = 0$.

4. (30) Let f_n be a sequence of functions defined on a set E, and suppose f is another function defined on E.

(a) Define what it means for the sequence f_n, to converge uniformly to f on the set E.

The sequence f_n converges uniformly to f on the set E if for all $\epsilon > 0$ there exist an N such that for $n > N$ and all $x \in E$ we have

$$|f_n(x) - f(x)| < \epsilon$$

(b) Let $f_n(x) = \dfrac{\ln(x/n)}{n}$, and set $f(x) = \lim\limits_{n\to\infty} f_n(x)$, with E equal to the open interval $(0, 1)$. That is, $f(x)$ is the pointwise limit of the sequence f_n. Determine f, and then determine whether or not the sequence of functions f_n converges uniformly to f on the set $E = (0, 1)$.

Taking the limit as $n \to \infty$ of the sequence f_n, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln(x/n)}{n} &= \lim\nolimits_{n\to\infty} \frac{(n/x)(-x/n^2)}{1} \\ &= \lim\nolimits_{n\to\infty} \frac{-1}{n} = 0 \end{aligned}$$

Thus, f_n converges to zero point wise on $(0, 1)$. However, this sequence does not converge to zero uniformly on this interval. The reason there is no uniformity in the convergence is that no matter how large we pick an n, once it is fixed we can let x approach zero until $\ln(x/n)$ is as large in absolute value as we want.s

5. (20) Suppose f_n converges uniformly to f on the set $E = [a,\, b]$. Assume also that each f_n is Riemann integrable on $[a,\, b]$, and that f is also Riemann integrable on $[a,\, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx\ .$$

Let $\epsilon > 0$. Pick N so large that for $n > N$ we have for all $x \in [a,\, b]$

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}\ .$$

Then for $n > N$ we have

$$\left| \int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx \right| = \left| \int_a^b [f_n(x) - f(x)]\ dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\ dx \leq \int_a^b \frac{\epsilon}{b-a}\ dx = \epsilon$$

Note: We assumed in this problem that the limit function f is Riemann integrable, which makes this result much easier to verify, since we don't have to prove that f is Riemann integrable.

6. (30) For each of the following power series, determine the radius and interval of convergence of the series.

a. $\sum_{n=1}^{\infty} \dfrac{x^n}{\ln(n+1)}$ b. $\sum_{k=0}^{\infty} \dfrac{(2x+1)^k}{3^k}$

(a) Taking the limit as n goes to infinity of the absolute value of the ratio of the $(n+1)^{\text{st}}$ term to the n^{th} term we have

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/\ln(n+2)}{x^n/\ln(n+1)} \right| = |x|\ .$$

Thus, the radius of convergence of this series is 1. At the endpoint 1 the series diverges since $\ln n < n$ and we know the harmonic series, whose terms are $1/n$ diverges. At the point -1 the series converges by the alternating series test. Thus, the interval of convergence is $[-1,\, 1)$.

(b) Taking the limit of the ratio of the terms as we did for the previous series we have

$$\lim_{k\to\infty} \left| \frac{(2x+1)^{k+1}/3^{k+1}}{(2x+1)^k/3^k} \right| = \left| \frac{2x+1}{3} \right|\ .$$

Thus, the radius of convergence is $3/2$. That is, x must satisfy the inequalities

$$\left| \frac{2x+1}{3} \right| < 1 \implies |2x+1| < 3 \implies \left| x + \frac{1}{2} \right| < \frac{3}{2}$$

The endpoints of the interval of convergence are -2 and 1. At $x = -2$ the terms of the series are ± 1 and at $x = 1$ the terms are all equal to 1. Thus, the series does not converge at either endpoint, so the interval of convergence is the open interval $(-2,\, 1)$.

7. (10) Let $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k5^k}$. Find a closed form expression for $f(x)$, and use it to determine the value of $f(1)$. Be sure to determine the interval of convergence of this series.

The radius of convergence of the series is 5, and the interval of convergence is $[-5, 5)$. To find a closed form expression for this series we note that each of the summands can be written as and integral.

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{x^k}{k5^k} = \sum_{k=1}^{\infty} \frac{1}{5^k} \int_0^x t^{k-1}\, dt = \int_0^x \left(\frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{t}{5}\right)^{k-1} \right) dt \\ &= \int_0^x \left(\frac{1}{5} \sum_{k=0}^{\infty} \left(\frac{t}{5}\right)^{k} \right) dt = \frac{1}{5} \int_0^x \frac{1}{1 - t/5}\, dt \\ &= -\ln(1 - x/5) \end{aligned}$$

Thus, $f(1) = -\ln(1 - 1/5) = \ln(5/4)$.