

1. (20)

(a) Define the statement,  $f(x)$  is continuous at  $a \in E$ .

This means that  $\forall \epsilon > 0, \exists \delta > 0$  such that if  $x \in E$  and  $|x-a| < \delta$ , then  $|f(x)-f(a)| < \epsilon$ .

(b) Let  $A$  be a set. Define  $\sup(A)$ .

The supremum of any set  $A$  is a number  $s$ , which is an upper bound of  $A$ , with the property that  $s$  is less than or equal to all upper bounds of the set  $A$ .

(c) Define the statement,  $\lim_{x \rightarrow a^+} f(x) = 6$ .

This means that  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $0 < x - a < \delta$  then  $|f(x) - 6| < \epsilon$ .

(d) State the completeness property of real numbers.

Every set of real numbers that is bounded above has a least upper bound.

2. (10) Using the definition of limit prove that

$$\lim_{x \rightarrow 2} (x^2 + 6) = 10.$$

Let  $\epsilon > 0$ , set  $\delta = \min\{1, \frac{\epsilon}{3}\}$  Then if

$|x - 2| < \delta$  we have  $|x + 2| < 3$  and

$$|x^2 + 6 - 10| = |x + 2||x - 2| < 3|x - 2| < \epsilon.$$

3. (15) Let  $A$  be a bounded set of real numbers. Let  $s = \sup(A)$ . Show there exists a sequence  $\{x_n\}$  of points in  $A$  such that  $\lim_{n \rightarrow \infty} x_n = s$ .

Since  $s$  is the least upper bound of  $A$  we know that for any  $\epsilon > 0$  there must be an element  $x \in A$  such that  $s - \epsilon < x \leq s$ . For each natural number  $n$  let  $x_n$  be an element of  $A$  such that  $s - \frac{1}{n} < x_n \leq s$ . Then  $\lim_{n \rightarrow \infty} x_n = s$ .

4. (20)

- (a) State the monotone convergence theorem for an increasing sequence and prove it.

Let  $x_n$  be an increasing sequence, which is bounded above. Then the limit of this sequence, as  $n$  goes to infinity, exists.

To see that this is true, let  $l = \sup(\{x_n\}_{n=1}^{\infty})$ . Then for any  $\epsilon > 0$  there is an  $N$  such that  $l - \epsilon < x_N \leq l$ , and since the sequence  $x_n$  is increasing this means that for every  $n > N$  we have  $l - \epsilon < x_N \leq x_n \leq l$ . Thus,

$$\lim_{n \rightarrow \infty} x_n = l.$$

- (b) Let  $0 < x < 1$ . Show that  $\lim_{n \rightarrow \infty} x^n = 0$ . You may not use calculus for this, nor logarithms either, just elementary properties of the real numbers.

Since  $0 < x < 1$ , we have  $0 < x^{n+1} < x^n$ . That is, the sequence  $x^n$  is decreasing and bounded below by 0. Thus, by the monotone convergence theorem, the limit as  $n \rightarrow \infty$  of  $x_n$  exists. Call this limiting value  $l$ . Then  $l$  must satisfy the inequalities  $0 \leq l < x < 1$ . The following shows that  $l = 0$ .

$$l = \lim_{n \rightarrow \infty} x^{n+1} = x \lim_{n \rightarrow \infty} x^n = xl.$$

This leads to the equation  $l(1 - x) = 0$ . Since  $1 - x \neq 0$ , we must have  $l = 0$ .

5. (15) Determine whether or not each of the following is true or false. If true give a proof, and if false supply a counter example:

- (a) If  $f$  is continuous on the interval  $(0, 1)$ , and  $\{x_n\}$  is a Cauchy sequence of points from the interval  $(0, 1)$ , then  $\{f(x_n)\}$  is also a Cauchy sequence.

This is false. A counter example is given by setting  $f(x) = 1/x$  and  $x_n = 1/n$ . Since  $x_n$  converges to 0, we know it is Cauchy, and the sequence  $f(x_n) = n$  is not Cauchy.

- (b) If  $x_n < a$  and  $\lim_{n \rightarrow \infty} x_n = l$ , then  $l < a$ .

This is false. Set  $x_n = 1 - 1/n$ . Then  $x_n < 1$ , but  $\lim_{n \rightarrow \infty} x_n = 1$ .

- (c) If  $I_n = (a_n, b_n)$  is a sequence of open nested intervals, i.e.,  $I_{n+1} \subset I_n$ , then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

This too is false. Set  $I_n = (0, 1/n)$ . Then  $I_{n+1} \subset I_n$ , and  $\bigcap_{n=1}^{\infty} I_n = \emptyset$

6. (10) Let  $f(x) = \begin{cases} x - 2, & x < 1 \\ x^2 - 1, & 1 \leq x \leq 2 \\ 5 - x, & 2 < x \end{cases} .$

(a) Prove or disprove that  $\lim_{x \rightarrow 1^-} f(x) = f(1)$ .

This is false.  $f(1) = 0$ , but

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x - 2) = -1.$$

Thus,  $f$  is not continuous from below at  $x = 1$ .

(b) Prove or disprove that  $f$  is continuous at  $1 \in E = [1, \infty)$ .

This is true. Since  $E = [1, \infty)$ , the limit as  $x \rightarrow 1$  with  $x \in E$  is the same as the limit as  $x$  approaches 1 from above, and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 - 1) = 0 = f(1).$$

7. (10) Let  $A$  and  $B$  be two disjoint countably infinite sets. Show that  $A \cup B$  is countably infinite.

Since  $A$  and  $B$  are countable infinite there are one-to-one functions  $f$  and  $g$  mapping the natural numbers  $N$  onto  $A$  and  $B$  respectively. Define the function  $h$  from  $N$  onto  $A \cup B$  as follows

$$h(n) = \begin{cases} f(\frac{n}{2}), & n = 2k \\ g(\frac{n+1}{2}), & n = 2k - 1 \end{cases} .$$

As  $n$  goes through all of the even integers,  $n/2$  goes through the integers in a one-to-one fashion, and as  $n$  goes through all of the odd integers  $(n - 1)/2$  goes through all of the integers in a one-to-one manner also. Clearly the codomain of  $h$  is  $A \cup B$ . Since  $f$  and  $g$  are onto  $A$  and  $B$  respectively,  $h$  is onto  $A \cup B$ . Moreover since both  $f$  and  $g$  are one-to-one, and  $A$  and  $B$  are disjoint  $h$  is also one-to-one.