

1. (20) Define the following:

(a) open set

A set $A \subseteq E^n$ is said to be open if $\forall \vec{x} \in A, \exists \delta > 0$ such that

$$B(\vec{x}, \delta) \subseteq A,$$

where $B(\vec{x}, \delta) = \{\vec{y} : \|\vec{x} - \vec{y}\| < \delta\}$.

(b) $f : D \rightarrow E^1$ is continuous at the point $\vec{x}_0 \in D$, where D is an open subset of E^n

f is continuous at the point \vec{x}_0 if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\text{if } \|\vec{x} - \vec{x}_0\| < \delta, \text{ then } |f(\vec{x}) - f(\vec{x}_0)| < \epsilon.$$

(c) Let $A \subseteq E^n$. What is the frontier of A ?

The frontier of A is the set of all frontier points of A ; a point \vec{x} is a frontier point of A if every neighborhood of \vec{x} contains at least one point of A and at least one point of A^c .

(d) Let $A \subseteq E^n$. What is an accumulation point of A ?

\vec{x} is an accumulation point of A if every neighborhood of \vec{x} contains an infinite number of points of A .

2. (20) State the Bolzano-Weierstrass theorem, and prove that if $A_n, n = 1, 2, \dots$ is a sequence of closed non-empty decreasing sets ($A_{n+1} \subseteq A_n$) with $\lim_{n \rightarrow \infty} \text{diam}(A_n) = 0$, then

$$\bigcap_{n=1}^{\infty} A_n = \{\vec{x}_0\}.$$

That is, the intersection of all the A_n consists of exactly one point.

The Bolzano-Weierstrass theorem says that every bounded infinite subset of E^n has an accumulation point.

To show that the intersection of the A_n is non-empty it suffices to assume that each of the A_n is infinite; for if one of them, say A_N is finite, then for all $n > N$, A_n must also be finite. If $A_N = \{a_1, a_2, \dots, a_k\}$, then one of these a_i 's must be in each of the A_n and hence in the intersection. So assuming that each A_n is infinite pick an element \vec{x}_1 from A_1 , then pick \vec{x}_2 from A_2 not equal to \vec{x}_1 , continue in this fashion. That is, pick \vec{x}_n from A_n such that $\vec{x}_n \notin \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{n-1}\}$. Set $A = \{\vec{x}_n\}_{n=1}^{\infty}$. Then A is infinite and bounded. Let \vec{x}_0 be an accumulation point (apply the Bolzano-Weierstrass theorem) of A . Then \vec{x}_0 must be an accumulation point of each of the A_n , and since each of these sets is closed we have $\vec{x}_0 \in A_n$ for each n . Thus, $\vec{x}_0 \in \bigcap_{n=1}^{\infty} A_n$.

The last thing to verify is that this intersection consists of at most one point. So suppose \vec{x} and \vec{y} both belong to the intersection. Then for each n we have

$$\|\vec{x} - \vec{y}\| \leq \text{diam}(A_n),$$

but the diameters of the A_n are converging to zero. Thus, we must have $\|\vec{x} - \vec{y}\| = 0$, or $\vec{x} = \vec{y}$.

3. (10) Let S be a compact subset of E^n .

(a) Define what compact means.

A set is said to be compact if every open cover of the set has a finite subcollection that is also a cover of the set.

(b) Show that S must be closed and bounded.

To see that S must be bounded, we note that $B(\vec{0}, n)$ is an open set and that $S \subseteq \cup_{n=1}^{\infty} B(\vec{0}, n)$. Compactness means that a finite subcollection of these $B(\vec{0}, n)$ must cover S . But then for some N we must have

$$S \subseteq \cup_{n=1}^N B(\vec{0}, n) = B(\vec{0}, N) .$$

Hence, S is bounded.

To see that S is closed suppose that \vec{x} is any point not in S . Then the sets $G_n = \{\vec{y} : \|\vec{x} - \vec{y}\| > 1/n\}$ are open for each n and

$$S \subseteq \cup_{n=1}^{\infty} G_n .$$

Compactness implies there is an N such that

$$S \subseteq \cup_{n=1}^N G_n = G_N .$$

Thus, any point not in S is an interior point of S^c . That is, S^c must be open, which means S is closed.

4. (10) Show that $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. Is the same equality true if intersection is replaced with union?

To see that we have equality suppose $\vec{x} \in \text{int}(A \cap B)$, then for some $\delta > 0$, $B(\vec{x}, \delta) \subset (A \cap B)$. But, this says $B(\vec{x}, \delta) \subset A$ and $B(\vec{x}, \delta) \subset B$. That is, \vec{x} is an interior point of both A and B . Hence $\vec{x} \in \text{int}(A) \cap \text{int}(B)$. Now suppose that $\vec{x} \in \text{int}(A) \cap \text{int}(B)$. Then there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that

$$B(\vec{x}, \delta_1) \subseteq A \text{ and } B(\vec{x}, \delta_2) \subseteq B .$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $B(\vec{x}, \delta) \subseteq A \cap B$, and \vec{x} is an interior point of $A \cap B$. Thus, the two sets are equal.

To see that equality does not hold for unions, let $A = (0, 1)$ and $B = [1, 2]$. Then

$$\begin{aligned} \text{int}(A \cup B) &= (0, 2) \text{ while} \\ \text{int}(A) \cup \text{int}(B) &= (0, 1) \cup (1, 2) , \end{aligned}$$

and the two sets are not equal. One can show that for arbitrary A and B that

$$\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B) ,$$

and, as the above example shows, this is the best we can do.

5. (20) For each of the following sets A determine if it is open or closed. Then for each of these sets find its interior, frontier, closure, exterior, and its set of accumulation points.

(a) $A = \{\vec{x} = (x^1, x^2) : x^1 < 2 - x^2\}$

The set A is open. To see this define $f : E^2 \rightarrow E^1$ by $f(x^1, x^2) = 2 - x^2 - x^1$. Then f is continuous on E^2 and the set $A = f^{-1}((0, \infty))$. Since $(0, \infty)$ is an open set and f is continuous, A must also be open.

- i. $\text{int}(A) = A$, since A is open
- ii. $\text{fr}(A) = \{(x^1, x^2) : x^1 = 2 - x^2\}$. The easiest way to see this is to note that if $\vec{x} = (x^1, x^2)$ is such that $x^1 = 2 - x^2$, then given any neighborhood about \vec{x} of radius δ there are points in this ball in A and in A^c . For example, consider a point of the form $\vec{x}_{\pm} = (x^1 \pm \delta/2, x^2)$. Both \vec{x}_+ and \vec{x}_- are in this ball with \vec{x}_+ in A^c and \vec{x}_- in A . Any point $\vec{x} = (x^1, x^2)$ for which $x^1 \neq 2 - x^2$ is either in A or in the set $\{\vec{x} = (x^1, x^2) : x^1 > 2 - x^2\}$. Since A is open, no point of A can be a frontier point, and since the second set is also open (same reason that A is open), none of its points can belong to the frontier of A .
- iii. $\text{cl}(A) = \{(x^1, x^2) : x^1 \leq 2 - x^2\}$.
- iv. $\text{ext}(A) = \{(x^1, x^2) : x^1 > 2 - x^2\}$. This follows from the fact that $\text{ext}(A) = E^2 - \text{cl}(A)$.
- v. The set of accumulation points of A is $\text{cl}(A)$. This follows from the fact that A has no isolated points.

(b) $A = \{(0, y) : 0 \leq y \leq 1\} \cup \{(x, \sin(1/x)) : 0 < x \leq 1\}$.

This set is neither open nor closed. Points of the form $(0, y)$ for $-1 \leq y < 0$ are frontier points of A , but are not in A so A cannot be closed. To see that A cannot be open, note that the point $(1, \sin 1) \in A$ but points of the form $(1 + \delta, \sin 1)$ are not in A yet points of this form are in any ball about $(1, \sin 1)$.

- i. $\text{int}(A) = \emptyset$
- ii. $\text{fr}(A) = \{(0, y) : -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) : 0 < x \leq 1\}$
- iii. $\text{cl}(A) = \text{fr}(A)$
- iv. $\text{ext}(A) = \text{cl}(A)^c$
- v. the set of accumulation points of A is the same as the closure of A .

6. (10) Let A be a subset of E^n . Define the distance of a point $x \in E^n$ to the set A as

$$\text{dist}(x, A) = \inf_{y \in A} \|x - y\| .$$

Show that there is a sequence x_n of points in A such that

$$\lim_{n \rightarrow \infty} \|x - x_n\| = \text{dist}(x, A) .$$

Since $\text{dist}(x, A) = \inf_{y \in A} \|x - y\|$, for each integer i there must be a point $x_i \in A$ such that $\|x - x_i\| < \text{dist}(x, A) + 1/i$. Thus we have

$$\text{dist}(x, A) = \inf_{y \in A} \|x - y\| \leq \|x - x_i\| < \text{dist}(x, A) + \frac{1}{i} .$$

This inequality implies that $\text{dist}(x, A) = \lim_{i \rightarrow \infty} \|x - x_i\|$.

7. (10) Let $f(x, y) = \frac{y}{x}$.

(a) Calculate $\lim_{(x,y) \rightarrow (2,1)} f(x, y)$

The calculation of the limit is easy and we get

$$\lim_{(x,y) \rightarrow (2,1)} \frac{y}{x} = \frac{1}{2}.$$

(b) Using the definition of limit verify your answer to part a.

Let $\epsilon > 0$, set $\delta = \min \left\{ 1, \frac{2\epsilon}{3} \right\}$. Note: if $\|(x, y) - (2, 1)\| < \delta$, then

$$|x - 2| \leq \|(x, y) - (2, 1)\| < 1 \implies \frac{1}{|x|} < 1$$

Thus

$$\begin{aligned} \left| \frac{y}{x} - \frac{1}{2} \right| &= \left| \frac{2y - x}{2x} \right| = \left| \frac{2y - 2 + 2 - x}{2x} \right| \\ &= \frac{1}{2|x|} |2(y - 1) + (2 - x)| \\ &< |y - 1| + \frac{|2 - x|}{2} \\ &\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$