1. (30) In each part of this question f denotes a mapping from $\triangle$ into E^1 where $\triangle$ is an open subset of E^n.

   (a) What does it mean to say that f is differentiable at some $t_0 \in \triangle$?

   f is differentiable at t_0 if there is a linear transformation L from E^n to E^1 such that

$$\lim_{\vec{h} \to \vec{0}} \frac{f\left(\vec{t}_0 + \vec{h}\right) - f\left(\vec{t}_0\right) - L\left(\vec{h}\right)}{\left|\vec{h}\right|} = 0$$

   (b) Use the definition of the differential of f at t_0 to compute $df\,(t_0) \cdot (5\vec{e}_1)$.

   Since f is differentiable at t_0 we must have

$$\lim_{h \to 0} \frac{f\left(\vec{t}_0 + h5\vec{e}_1\right) - f\left(\vec{t}_0\right) - L\left(h5\vec{e}_1\right)}{5\left|h\right|} = 0,$$

   where h is a real number and $h5\vec{e}_1$ is the $\vec{h}$ in part a. Dropping the absolute value sign in the denominator, and using the fact that L is linear, we have

$$\begin{aligned} \lim_{h \to 0} \frac{f\left(\vec{t}_0 + h5\vec{e}_1\right) - f\left(\vec{t}_0\right) - h5L\left(\vec{e}_1\right)}{5h} &= 0 \\ \lim_{h \to 0} \frac{f\left(\vec{t}_0 + h5\vec{e}_1\right) - f\left(\vec{t}_0\right)}{5h} &= L\left(\vec{e}_1\right) \ . \end{aligned}$$

   Noticing that the left hand side is the partial derivative of f with respect to t^1, we have $f_1\left(t_0\right) = L\left(\vec{e}_1\right)$. Thus,

$$df\left(t_0\right) \cdot \left(5\vec{e}_1\right) = L\left(5\vec{e}_1\right) = 5L\left(\vec{e}_1\right) = 5f_1\left(t_0\right) \ .$$

   (c) Show that if f is differentiable at t_0, then f is continuous at t_0.

   Let $\epsilon > 0$. Since f is differentiable at t_0 there is a $\delta > 0$ such that if $h \in B\left(0, \delta\right)$, then

$$\begin{aligned} \frac{\left|f\left(\vec{t}_0 + \vec{h}\right) - f\left(\vec{t}_0\right) - L\left(\vec{h}\right)\right|}{\left|\vec{h}\right|} &< \epsilon \\ \left|f\left(\vec{t}_0 + \vec{h}\right) - f\left(\vec{t}_0\right) - L\left(\vec{h}\right)\right| &< \epsilon\left|\vec{h}\right| \\ \left|f\left(\vec{t}_0 + \vec{h}\right) - f\left(\vec{t}_0\right)\right| - \left|L\left(\vec{h}\right)\right| &< \epsilon\left|\vec{h}\right| \\ \left|f\left(\vec{t}_0 + \vec{h}\right) - f\left(\vec{t}_0\right)\right| &< \left|L\left(\vec{h}\right)\right| + \epsilon\left|\vec{h}\right| \\ &\leq \left(\left|L\right| + \epsilon\right)\left|\vec{h}\right| \\ &\leq \left(\left|L\right| + \epsilon\right)\epsilon \end{aligned}$$

   Since we can make ϵ as small as we wish, we see that f is continuous at t_0.

2. (10) Let $f(x,\, y)$ be a real valued function that belongs to $C^1\left(R^2\right)$. Show that f is differentiable at every point of R^2.

We want to show that for any $(x_0, y_0) \in R^2$ we have

$$\lim_{(x,y)\,\to\,(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_1|_{(x_0,y_0)}(x-x_0) - f_2|_{(x_0,y_0)}(y-y_0)}{|(x-x_0, y-y_0)|} = 0$$

Since f is C^1 everywhere we can use the one-dimensional version of the mean value theorem. This gives us

$$\begin{aligned}
&\lim_{(x,y)\,\to\,(x_0,y_0)} \left| \frac{f(x,y) - f(x_0,y_0) - f_1|_{(x_0,y_0)}(x-x_0) - f_2|_{(x_0,y_0)}(y-y_0)}{|(x-x_0, y-y_0)|} \right| \\
= &\lim_{(x,y)\,\to\,(x_0,y_0)} \left| \frac{f(x,y) - f(x_0,y) + f(x_0,y) - f(x_0,y_0) - f_1|_{(x_0,y_0)}(x-x_0) - f_2|_{(x_0,y_0)}(y-y_0)}{|(x-x_0, y-y_0)|} \right| \\
= &\lim_{(x,y)\,\to\,(x_0,y_0)} \left| \frac{f_1|_{(\xi,y)}(x-x_0) + f_2|_{(x_0,\eta)}(y-y_0) - f_1|_{(x_0,y_0)}(x-x_0) - f_2|_{(x_0,y_0)}(y-y_0)}{|(x-x_0, y-y_0)|} \right| \\
= &\lim_{(x,y)\,\to\,(x_0,y_0)} \left| \frac{\left[f_1|_{(\xi,y)} - f_1|_{(x_0,y_0)}\right](x-x_0) + \left[f_2|_{(x_0,\eta)} - f_2|_{(x_0,y_0)}\right](y-y_0)}{|(x-x_0, y-y_0)|} \right| \\
\leq &\lim_{(x,y)\,\to\,(x_0,y_0)} \left| f_1|_{(\xi,y)} - f_1|_{(x_0,y_0)} \right| + \left| f_2|_{(x_0,\eta)} - f_2|_{(x_0,y_0)} \right|
\end{aligned}$$

The numbers $\xi$and η lie between x and x_0 and y and y_0 respectively. Thus, as (x, y) approaches (x_0, y_0), ξ approachs x_0 and y_0 approaches η. Since the partial derivatives f_1 and f_2 are continuous at (x_0, y_0), both of the above two terms go to zero.

3. (25) Let $g(x,y,z) = \left(g^1, g^2\right)$, where $g^1(x,y,z) = x^2 + y^2 - z^2$ and $g^2(x,y,z) = x^3 - 2y^2 + 4z$.

(a) Show that g is differentiable at all points of E^3.

Since both of g's component functions are $C^\infty\left(E^2\right)$ they are differentiable at all points of E^2. Thus, g itself is differentiable everywhere.

(b) Since $g(1,2,1) = (4,-3)$, we know that $x = 1$, $y = 2$, and $z = 1$ is a solution to the following system of two equations in three unknowns

$$\begin{aligned} g^1(x,y,z) &= 4 \\ g^2(x,y,z) &= -3\ . \end{aligned}$$

What does the implicit function theorem tell us about being able to solve this system for some of the unknowns (be specific about which variables can be solved for) in terms of the others? Be sure to verify that g satisfies the conditions of the implicit function theorem.

The differential Dg of g evaluated at the point $(1,2,1)$ equals

$$Dg|_{(1,2,1)} = \begin{bmatrix} 2 & 4 & -2 \\ 3 & -8 & 4 \end{bmatrix}\ .$$

The rank of this matrix is 2, the maximal rank of Dg. The pairs of columns, $\{1,2\}$ and$\{1,3\}$ are linearly independent, and the pair $\{2,3\}$ is linearly dependent. The fact that columns 1 and 2 are linearly independent implies that we can solve the equations for x and y in terms of z, and the independence of columns 1 and 3 imply that we can solve for x and z in terms of y in some interval about the point $y = 2$. Moreover the functions that arise from solving this system of equations will be as smooth as g. That is, they will be C^∞ on their domains of definition.
The fact that columns 2 and 3 are dependent indicates that there may be some problems if we try to solve for y and z in terms of x.

(c) Suppose that the above system is solved for x and z in terms of y in some neighborhood of $(1,2,1)$. Let $X(y)$, and $Z(y)$ denote these solutions. That is, $X(2) = 1$, $Z(2) = 1$ and

$$\begin{aligned} g^1(X(y), y, Z(y)) &= 4 \\ g^2(X(y), y, Z(y)) &= -3\ , \end{aligned}$$

for y in some interval containing the value 2. Determine $X'(2)$ and $Z'(2)$.

Differentiating each of these equations with respect to y leads to the equations

$$\begin{aligned} g^1_1(X(y), y, Z(y))\, X' + g^1_2(X(y), y, Z(y)) + g^1_3(X(y), y, Z(y))\, Z' &= 0 \\ g^2_1(X(y), y, Z(y))\, X' + g^2_2(X(y), y, Z(y)) + g^2_3(X(y), y, Z(y))\, Z' &= 0\ . \end{aligned}$$

Evaluating at the point $(1,2,1)$ we have the system

$$\begin{aligned} 2X' - 2Z' &= -4 \\ 3X' + 4Z' &= 8\ . \end{aligned}$$

Notice the coefficient matrix consists of columns 1 and 3 of Dg. Since they are linearly independent we can solve this system and get

$$\begin{aligned} X'(2) &= 0 \\ Z'(2) &= 2\ . \end{aligned}$$

(d) What happens if we try to solve for y and z in terms of x ? That is, if $Y(x)$, and $Z(x)$ are such that $Y(1) = 2$, $Z(1) = 1$, and

$$\begin{aligned} g^1(x, Y(x), Z(x)) &= 4 \\ g^2(x, Y(x), Z(x)) &= -3 , \end{aligned}$$

what happens when we try to determine $Y'(1)$ and $Z'(1)$?

Differentiating the equations with respect to x gives the equations

$$\begin{aligned} g_1^1(x, Y(x), Z(x)) + g_2^1(x, Y(x), Z(x))\, Y' + g_3^1(x, Y(x), Z(x))\, Z' &= 0 \\ g_1^2(x, Y(x), Z(x)) + g_2^2(x, Y(x), Z(x))\, Y' + g_3^2(x, Y(x), Z(x))\, Z' &= 0 . \end{aligned}$$

Evaluating at the point $(1, 2, 1)$ we have

$$\begin{aligned} 4Y' - 2Z' &= -2 \\ -8Y' + 4Z' &= -3 \end{aligned}$$

Notice that this time the coefficient matrix, columns 2 and 3 of Dg, is not invertible. Even worse, this system has no solution.

4. (20) Let $f(x, y, z) = xy^2 - 3x + z^2$ with domain equal to E^3.

(a) Does f have a global maximum or global minium?

f does not have any global extremum. To see that there is no global extrema, notice that $f(0, 0, z) = z^2$, and this can be made arbitrarily large, and $f(x, 1, 0) = -2x$ can be made arbitrarily small.

(b) Find all critical points of f.

$$df = \left(y^2 - 3, 2xy, 2z\right) .$$

Thus $df = \vec{0}$ implies $x = z = 0$ and $y = \pm\sqrt{3}$.

(c) Characterize the critical points of f. That is, are they local extrema (if yes, what kind) or not?

Computing the Hessian matrix Q of f we have

$$Q(x, y, z) = \begin{bmatrix} 0 & 2y & 0 \\ 2y & 2x & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

Evaluating at the critical points we have

$$Q\left(0, \pm\sqrt{3}, 0\right) = \begin{bmatrix} 0 & \pm 2\sqrt{3} & 0 \\ \pm 2\sqrt{3} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

The characteristic polynomial of this matrix is

$$(2 - \lambda)\left(\lambda^2 - 12\right) .$$

Thus, the eigenvalues of Q at each of its two critical points are $\pm\sqrt{12}$ and 2. Since the eigenvalues are both positive and negative at each critical point, the critical points are not local extrema, they are saddle points.

5. (10) Let f denote a mapping from $\triangle$ into E^n where $\triangle$ is some open subset of E^n. Suppose that $f \in C^1(\triangle)$, and $Jf(t_0) = \det\left(Df|_{t_0}\right)$, the Jacobian of f at t_0, is not equal to zero. Show there is a neighborhood Ω of t_0 such that f is one-to-one on Ω.

Let A be the $n \times n$ matrix whose rows are the differentials of the component functions of f, but each row is evaluated at a different point t_i. That is,

$$A(t_1, t_2, \cdots, t_n) = \begin{bmatrix} df^{\,1}(t_1) \rightarrow \\ df^{\,2}(t_2) \rightarrow \\ \vdots \\ df^{\,n}(t_n) \rightarrow \end{bmatrix}$$

Set $DD(t_1, t_2, \cdots, t_n) = \det(A)$. Then, since $f \in C^1(\triangle)$, DD is a continuous function of the n^2 variables t_i^j, and $DD(t_0, t_0, \cdots, t_0) \neq 0$. This inequality follows from the fact that $DD(t_0, t_0, \cdots, t_0) = Jf(t_0) \neq 0$. Thus, there is a $\delta > 0$ such that if $|t_i - t_0| < \delta$ for each i, then $DD(t_1, t_2, \cdots, t_n) \neq 0$. Now suppose that x_1 and x_2 are in this ball of radius δ about the point t_0. Using the mean value theorem, we have

$$\begin{aligned} f(x) - f(y) &= \begin{bmatrix} f^{\,1}(x) - f^{\,1}(y) \\ f^{\,2}(x) - f^{\,2}(y) \\ \vdots \\ f^{\,n}(x) - f^{\,n}(y) \end{bmatrix} = \begin{bmatrix} df^{\,1}(\xi_1) \cdot (x-y) \\ df^{\,2}(\xi_2) \cdot (x-y) \\ \vdots \\ df^{\,n}(\xi_n) \cdot (x-y) \end{bmatrix} \\ &= \begin{bmatrix} df^{\,1}(\xi_1) \rightarrow \\ df^{\,2}(\xi_2) \rightarrow \\ \vdots \\ df^{\,n}(\xi_n) \rightarrow \end{bmatrix} \begin{bmatrix} x^1 - y^1 \\ x^2 - y^2 \\ \vdots \\ x^n - y^n \end{bmatrix}, \end{aligned}$$

where each ξ_i lies between x and y, and hence in the ball about t_0 of radius δ.

The $n \times n$ matrix in the above line is $A(\xi_1, \xi_2, \cdots, \xi_n)$ and its determinant is $DD(\xi_1, \xi_2, \cdots, \xi_n)$, which is not equal to zero. Thus, if $f(x) - f(y) = 0$ we must have $x^i = y^i$ for each i, or $x = y$. Thus, f is one-to-one on the ball of radius δ centered at t_0.

6. (5) Suppose $f : E^2 \rightarrow E^2$ is such that $f(2,1) = (-2,3)$, and $Df|_{(2,1)} = \begin{bmatrix} 5 & -3 \\ 7 & -1 \end{bmatrix}$. Estimate the value of $f(2.1, 0.95)$.

$$\begin{aligned} f(2.1, 0.95) &\approx f(2,1) + Df|_{(2,1)} \begin{bmatrix} 0.1 \\ -0.05 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 & -3 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.05 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0.65 \\ 0.75 \end{bmatrix} \\ &= \begin{bmatrix} -1.35 \\ 3.75 \end{bmatrix} \end{aligned}$$