Math 410 Final Exam Solutions December 3, 2004

1. Let $f^{\,1}$ and $f^{\,2}$ be two continuous real valued functions defined on an open set $\triangle \subseteq E^n$. Set $g(t) = \max\left\{f^{\,1}(t), f^{\,2}(t)\right\}$ for all $t \in \triangle$. Must g be a continuous function?

   The answer is yes. Let $t_0 \in \triangle$ be an arbitrary point in the domain of the two given functions. Let $\epsilon > 0$. We may suppose with out lose of generality that $f^{\,1}(t_0) \geq f^{\,2}(t_0)$. Suppose $f^{\,1}(t_0) > f^{\,2}(t_0)$. Set

$$\hat{\epsilon} = \frac{f^{\,1}(t_0) - f^{\,2}(t_0)}{3} \ .$$

   Then $f^{\,2}(t_0) + \hat{\epsilon} < f^{\,1}(t_0) - \hat{\epsilon}$. Thus, there is a $\delta > 0$ such that if $|t - t_0| < \delta$ then

$$f^{\,2}(t) < f^{\,2}(t_0) + \hat{\epsilon} < f^{\,1}(t_0) - \hat{\epsilon} < f^{\,1}(t_0) \ .$$

   Thus, for $|t - t_0| < \delta$ we have $g(t) = f^{\,1}(t)$. Thus,

$$\lim_{t \to t_0} g(t) = \lim_{t \to t_0} g(t) = \lim_{t \to t_0} f^{\,1}(t) = f^{\,1}(t_0) = g(t_0) \ .$$

   The last case to consider is when $f^{\,1}(t_0) = f^{\,2}(t_0)$. But then for any $\epsilon > 0$, there is a $\delta > 0$ such that if $|t - t_0| < \delta$, we have

$$\begin{aligned} f^{\,1}(t_0) - \epsilon &= g(t_0) - \epsilon < f^{\,1}(t) < g(t_0) + \epsilon \\ f^{\,2}(t_0) - \epsilon &= g(t_0) - \epsilon < f^{\,2}(t) < g(t_0) + \epsilon \end{aligned}$$

   Clearly this implies that $\max\left\{f^{\,1}(t), f^{\,2}(t)\right\}$ also lies between the two values $g(t_0) \pm \epsilon$. Thus, we have

$$g(t_0) - \epsilon < g(t) < g(t_0) + \epsilon,$$

   and g is continuous at t_0.

2. Let $f : \triangle \to E^1$, where $\triangle$ is an open subset of E^n. Suppose that f is differentiable at $t_0 \in \triangle$, and $df(t_0) \neq \vec{0}$. Show that the following limit cannot exist.

$$\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{\|h\|}$$

   Since df is not zero, at least one of f's partial derivatives is not zero at t_0. Assume that f_1 is non-zero at t_0. That is,

$$\lim_{h \to 0} \frac{f(t_0 + he_1) - f(t_0)}{h} = f_1(t_0) \neq 0 \ .$$

   So if $\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{\|h\|}$ exists, it must be true that

$$\lim_{h \to 0} \frac{f(t_0 + he_1) - f(t_0)}{|h|}$$

   must also exist. Call the limiting value K. Then we have

$$\begin{aligned} K &= \lim_{h \to 0^+} \frac{f(t_0 + he_1) - f(t_0)}{|h|} = \lim_{h \to 0^+} \frac{f(t_0 + he_1) - f(t_0)}{h} = f_1 \\ K &= \lim_{h \to 0^-} \frac{f(t_0 + he_1) - f(t_0)}{|h|} = -\lim_{h \to 0^+} \frac{f(t_0 + he_1) - f(t_0)}{h} = -f_1 \end{aligned}$$

   This says that f_1 must equal zero, which contradicts that it is not zero.

3. Let $f : \triangle \to E^1$, where $\triangle$ is an open subset of E^n. Suppose that f is differentiable at $t_0 \in \triangle$. Explain why the linear transformation L acting on a vector v equals

$$L(v) = \sum_{i\,=\,1}^{n} f_i(t_0)\, v^{\,i} \ ,$$

where L satisfies

$$\lim_{v\,\to\,0} \frac{f(t_0 + v) - f(t_0) - L(v)}{|v|} = 0 \ ,$$

and $f_i(t_0)$ denotes the partial derivative of f with respect to t^i at the point t_0.

For any vector $v = \left(v^1,\ \cdots,\ v^n\right)$, we have

$$L(v) = L\left(\sum_{i\,=\,1}^{n} v^{\,i} e_i\right) = \sum_{i\,=\,1}^{n} v^{\,i} L(e_i) \ .$$

Thus, we need to show that $L(e_i) = f_i$. Using the definition of f being diffentiable at t_0 we have .

$$0 = \lim_{h\,\to\,0} \frac{f(t_0 + he_i) - f(t_0) - L(he_i)}{|he_i|} \ .$$

This implies that

$$0 = \lim_{h\,\to\,0} \frac{f(t_0 + he_i) - f(t_0) - L(he_i)}{h}$$

which in turn implies

$$\lim_{h\,\to\,0} \frac{f(t_0 + he_i) - f(t_0)}{h} = \lim_{h\,\to\,0} \frac{L(he_i)}{h} = L(e_i) \ .$$

Since the left hand side is the partial derivative of f with respect to x^i, we've shown that $L(e_i) = f_i$.

4. Let $f \in C\left(E^1\right)$ be a real valued function.

   (a) Let $\beta(x) = \begin{cases} c_1 & x < K \\ c_2 & K \le x \end{cases}$. Show that

$$\int_a^b f\ d\beta = \begin{cases} f(K)(c_2 - c_1) & \text{if } a < K \le b \\ 0 & \text{otherwise} \end{cases}$$

Let $P = \{x_0,\, x_1,\ \cdots,\ x_n\}$ be any partition of $[a,\, b]$. Suppose first that $K \notin (a, b]$. There are two possibilities either $K \le a$ or $K > b$. If $K \le a$ then $\beta(x_i) = c_2$ for every $x_i \in P$, and we have $\beta(x_i) - \beta(x_{i-1}) = c_2 - c_2 = 0$. Thus, if $K \le a$, we have $S(P, f, \beta) = 0$ for every partition P. Hence, $\int_a^b f\ d\beta = 0$. If $K > b$, then $\beta(x_i) = c_1$ for each $x_i \in P$. Hence, in this case, we also have $\int_a^b f\ d\beta = 0$.

Suppose next that $a < K \le b$. Let $P = \{x_0,\, x_1,\ \cdots,\ x_n\}$ be any partition of $[a,\, b]$ such that K is one of the partition points. This means we can write P as

$$P = \{x_0,\ \cdots x_{j-1},\, K,\, x_{j+1},\ \cdots,\, x_n\} \ .$$

That is, $K = x_j$ for some j between 1 and n. $K \neq x_0$, since $x_0 = a < K \leq b = x_n$. For such a partition,

$$\begin{aligned} S(P,\, f,\, \beta) &= \sum_{k\,=\,1}^{n} f(t_k)\left[\beta(x_k) - \beta(x_{k-1})\right] \\ &= \sum_{k\,=\,1}^{j-1} f(t_k) \triangle\beta_k + f(t_j)\left[\beta(K) - \beta(x_{j-1})\right] + \sum_{k\,=\,j+1}^{n} f(t_k)\triangle\beta_k \\ &= \sum_{k\,=\,1}^{j-1} f(t_k)(c_1 - c_1) + f(t_j)\left[c_2 - c_1\right] + \sum_{k\,=\,j+1}^{n} f(t_k)(c_2 - c_2) \\ &= f(t_j)\left[c_2 - c_1\right] \ . \end{aligned}$$

The function f is continuous at the point K. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that

$$\text{if } |x - K| < \delta, \text{ then } |f(x) - f(K)| < \frac{\epsilon}{1 + |c_2 - c_1|} \ .$$

Let P_ϵ be any partition of $[a, b]$ such that $K \in P$, and for each i we have $|x_i - x_{i-1}| < \delta$. Let P be any partition that is finer than P_ϵ. Then P contains K and the distance between any two of its points is less than δ. Thus,

$$\begin{aligned} |S(P,\, f,\, \beta) - f(K)(c_2 - c_1)| &= |f(t_j)(c_2 - c_1) - f(K)(c_2 - c_1)| \\ &= |c_2 - c_1|\,|f(t_{j+1}) - f(K)| \\ &< |c_2 - c_1| \frac{\epsilon}{1 + |c_2 - c_1|} < \epsilon. \end{aligned}$$

Remember $t_j \in [x_{j-1},\, K]$, which means $|t_j - K| < \delta$.

(b) Let $\alpha(x)$ denote the greatest integer function. That is, $\alpha(x) = n$, where n is that unique integer satisfying $n \leq x < n + 1$. Show that

$$\int_0^N f\ d\alpha = \sum_{k\,=\,1}^{N} f(k)$$

for every positive integer N.

Define the function

$$\alpha_j(x) = \begin{cases} 0 & x < j \\ 1 & j \leq x \end{cases} .$$

Then on the interval $[0, N]$ the greatest integer function α is equal to a sum of some of the α_j's.

$$\alpha(x) = \sum_{j=1}^{N} \alpha_j(x)$$

Thus, since a continuous function f is Stieltjes integrable with respect to each of the α_j's on the interval $[0,\, N]$, we know f is integrable on $[0,\, N]$ and

$$\begin{aligned} \int_0^N f\ d\alpha &= \int_0^N f\ d\left(\sum_{j=1}^{N} \alpha_j\right) = \sum_{j=1}^{N} \int_0^N f\ d\alpha_j \\ &= \\ &= \sum_{j=1}^{N} f(j) \end{aligned}$$