1. (10) A large basket of fruit contains 3 oranges, 2 apples, and 5 bananas. If a piece of fruit is chosen at random, what is the probability of getting an orange or a banana?

$$p = \frac{3+5}{10} = 0.8$$

2. (10) Three cards are chosen at random for a standard deck of 52 playing cards.

   (a) What is the probability of getting an ace, a king, and a queen in that order?

   $$p = \frac{4^3}{52 \times 51 \times 50} = \frac{8}{16,575} \approx 4.83 \times 10^{-4}$$

   (b) What is the probability of getting an ace, a king, and a queen if order does not matter?

   $$p = \frac{4^3}{\binom{52}{3}} = \frac{6 \times 4^3}{52 \times 51 \times 50} = \frac{16}{5525} \approx 2.896 \times 10^{-3}$$

3. (30) Two dice are rolled. Assume any side is equally likely to appear up. Let E be the event that the sum of the two numbers rolled is an 8, and let F be the event that one of the two numbers is a 3. Compute the following probabilities.

(a) $p(E)$

$$p(E) = \frac{5}{36}$$

(b) $p(F)$

$$p(F) = \frac{11}{36}$$

(c) $p(E \cap F)$

$$p(E \cap F) = \frac{2}{36} = \frac{1}{18}$$

(d) $p(E \cup F)$

$$\begin{aligned} p(E \cup F) &= p(E) + p(F) - p(E \cap F) \\ &= \frac{5}{36} + \frac{11}{36} - \frac{1}{18} \\ &= \frac{14}{36} = \frac{7}{18} \end{aligned}$$

(e) $p(F|E)$

$$p(F|E) = \frac{p(E \cap F)}{p(E)} = \frac{1/18}{5/36} = \frac{2}{5}$$

(f) Are the events E and F independent?

No, since $p(F|E) \neq p(F)$ the two events are not independent.

4. (15) A coin will turn up heads when tossed with probability p. What is the probability that exactly k heads will appear in n tosses? Your answer must include not only the probability, but also an explanation.

$$p\,(\text{k successes}) = \binom{n}{k} p^k (1-p)^{n-k}$$

One way to see that this is correct is to think of an n-tuple whose first k slots are filled with s's and whose last $(n-k)$ slots are filled with f's. This represents the event that the first k tosses were heads and the last $n-k$ tosses were tails. Since each toss is independent of the outcome of any other toss, the probability of this event is $p^k(1-p)$. As there are $\binom{n}{k}$ ways to put k s's and $(n-k)$ f's into an n-tuple the probability of exactly k successes must be the expression above.

5. (20) An urn contains 5 red and 9 black balls. Suppose three balls are picked one at a time without replacement. Let E be the event that a red ball was withdrawn on the second pick, and let F be the event that a black ball was withdrawn on the first pick.

   (a) What is the probability of E?

   Let r_i and b_i represent the event that a red or black ball was drawn on the i^{th} pick respectively. Then we have

$$p\,(E) = p\,(r_2) = p\,(r_1) = \frac{5}{14} \approx 0.357$$

   (b) What is the probability of F given E?

$$\begin{aligned} p\,(F|E) &= p\,(b_1|r_2) = \frac{p\,(r_2 b_1)}{p\,(r_2)} \\ &= \frac{p\,(r_2|b_1)\,p\,(b_1)}{p\,(r_2)} = \frac{(5/13)\,(9/14)}{5/14} \\ &= \frac{9}{13} \approx 0.692 \end{aligned}$$

6. (15) 18 identical marbles are to be placed in 6 different containers, with some containers possibly empty.

   (a) How many ways can this be done?

   This is question is equivalent to the problem of how many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$$

   The number of such solutions is

$$\binom{18+6-1}{6-1} = \binom{23}{5} = 33,649$$

   (b) If each possible placement is equally likely, what is the probability that exactly 2 marbles will be placed in three different containers?

   Some interpreted the question in such a way as to allow more than 3 containers to contain 2 marbles. Under that interpretation, which is not what was asked, the number of ways to place the marbles is $\binom{6}{3}$ times the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 = 12 \ ,$$

   which is $\binom{12+3-1}{3-1} = \binom{14}{2} = 91$. Hence under this reading of the problem the probability is

$$\frac{\binom{6}{3}91}{33,694} = \frac{1820}{33,694} \approx 0.054 \ .$$

   Now let's see what the probability is under the rule that exactly three containers contain exactly 2 marbles.
   We first pick the three containers which will each contain 2 marbles. We can do this in $\binom{6}{3}$ ways. Once these three are picked we next count the number of ways to put 12 marbles into 3 different containers in such a way that no container contains 2 marbles. We count this by subtracting from the number of ways we can place 12 marbles into 3 containers the number of ways we can do this with one container containing 2 marbles and the number of ways we can do this with two containers containing 2 marbles.

The number of ways we can put 12 marbles into 3 containers so that one container contains 2 marbles is

$$\binom{3}{1}\left(\begin{array}{c}\text{number of ways 10 marbles can} \\ \text{be placed in two containers so} \\ \text{that neither contains 2 marbles}\end{array}\right) = \binom{3}{1}(9) = 27$$

The number of ways we can put 12 marbles into 3 containers so that two containers contain 2 marbles is

$$\binom{3}{2} = 3$$

Thus, the number of ways we can put 18 marbles into 6 containers so that exactly three containers contain 2 marbles is

$$\binom{6}{3}(91 - 27 - 3) = 1220$$

Hence the probability of this event is

$$\frac{1220}{33,649} \approx 0.036$$