Math 412 Solutions to Exam 1 Oct. 3, 2002

1. (40) Consider the following problem

$$\frac{u}{t} \quad \frac{{}^2 u}{x^2}, \quad 0 \quad x \quad , \quad 0 \quad t$$

$$\frac{u\ 0,t}{x} \quad 0, \quad \frac{u \quad ,t}{x} \quad 0, \quad 0 \quad t$$

$$u\ x,0 \quad \begin{cases} x, & \text{if } 0 \quad x \quad \overline{2} \\ x, & \text{if } \overline{2} \quad x \end{cases}$$

a. Find a solution to the above partial differential equation, boundary values, and initial value.

Solutions of the partial differential equation of the form $u\ x,t \quad g\ x\ h\ t$ are first found. If there is such a solution then the functions g and h must satisfy the ordinary differential equations

$$\frac{g \quad x}{g\ x} \quad \frac{h \quad t}{h\ t} \qquad {}^2$$

Each of the two ratios must be constant as the g ratio is a function of x, while the h ratio is a function of t. The general solutions to these equations are:

$$g\ x \quad a\sin \quad x \quad b\cos \quad x \text{ and } h\ t \quad ce^{\ {}^2 t}, \text{ if } \quad 0.$$

If $\quad 0$, then $g\ x \quad ax \quad b$, and the boundary conditions force $g\ x \quad b$ for all x. In the case $\quad 0$, the boundary conditions imply that g must satisfy $g\ \ 0 \quad 0$ and $g \quad 0$. Thus, $g\ x \quad \cos\ nx$, where n is any nonnegative integer, and we have an infinite number of functions $u_n\ x,t \quad \cos\ nx\ e^{\ n^2 t}$, which satisfy the homogeneous partial differential equation and the two homogeneous boundary conditions. Linear combinations of these functions will also satisfy these three conditions. To find a solution , which also satisfies the initial condition $u\ x,0 \quad f\ x$, we try

$$u\ x,t \quad \frac{a_0}{2} \quad \underset{n\ \ 1}{\quad} a_n \cos\ nx\ e^{\ n^2 t}.$$

This leads to requiring the a_n to satisfy

$$f\ x \quad u\ x,0 \quad \frac{a_0}{2} \quad \underset{n\ \ 1}{\quad} a_n \cos\ nx$$

on the interval $0, \quad$. Thus, the a_n equal

$$a_n \quad \frac{2}{\quad} \underset{0}{\quad} f\ x\ \cos\ nx\ dx \quad \text{for } n \quad 0, 1, \quad .$$

b. Discuss why what you found in part a is a solution.

Since the partial differential equation is homogeneous and linear as are the boundary conditions, we expect the sum of an infinite number of functions which satisfy them to also satisfy them.

The function f is continuous and piecewise smooth on the interval $0,\ \ $. More over $f\,0\quad f\quad$. Thus, the even extension of f to the interval $\ ,\ $, and then the 2 periodic extension of the extended function will be continuous everywhere, and piecewise smooth on every interval. In fact, the derivative exists everywhere except at integer multiples of . Moreover it is easy to show that $|a_n| \quad c/n^2$ for some fixed constant c. This enables one to show that the series defining u converges uniformly to it on any time interval bounded away from $t \quad 0$. Similar arguments enable one to see that any partial derivative of the partial sums converge uniformly to something, and that something then has to be the corresponding partial derivative of u. The estimate on the a_n's also allow one to show that the limit as t approaches zero from above exists and equals f.

2. (30) Let $f\,x \quad x^2 \quad x$.

a. Find the cosine series expansion of f on the interval $0,2$.

$$f \quad \frac{a_0}{2} \quad \underset{n\ 1}{} \; a_n \cos\left(\frac{n\ \ x}{2}\right),$$

where

$$a_n \quad \int_0^2 f\,x\ \cos\left(\frac{n\ \ x}{2}\right) dx$$

$$\frac{12\cos\ n \qquad 4}{n^2\ \ ^2}$$

$$\frac{4 \quad 12 \quad 1\ \ ^n}{n^2\ \ ^2} \quad \text{for } n \quad 1, \quad ,$$

and $a_0 \quad \frac{2}{3}$.

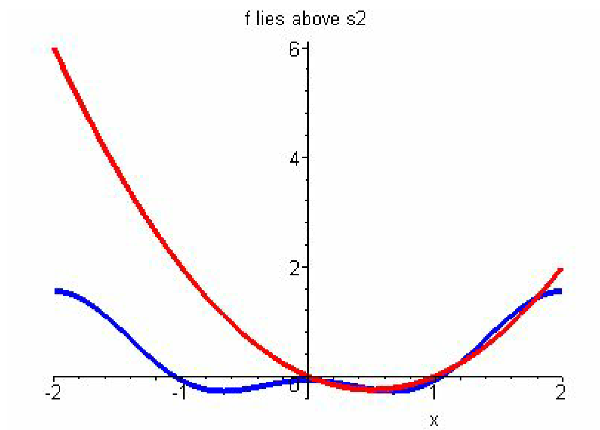
b. If $S_N\ x \quad \frac{a_0}{2} \quad \sum_{k\ 1}^{N} a_k \cos \frac{k}{2} x$ denote the partial sums of the cosine series expansion of f, what are the limiting values of $S_N\ x$ for each x in $0,2$? That is, for each x in $0,2$, what is

$$\lim_{N} S_N\ x\ ?$$

If we take the even extension of f to $\ 2,0$, and then the 4 periodic extension of that, we wind up with a function with is continuous everywhere. In particular for every x in the interval $0,2$, we have $f\,x \quad f\,x \qquad f\,x \quad$. Thus, $f\,x \qquad f\,x \qquad f\,x \quad /2$. Since f is piecewise smooth on $[0,2]$, its even extension to $[-2,0]$ is piecewise smooth on $[-2,2]$. Thus, for each x in $(0,2)$, we have

$$\lim_{N\to\infty} S_N(x) = f(x).$$

c. On the same graph plot $f(x)$ and $S_2(x)$ for x between -2 and 2. Explain why you get what you get. Note: your plots do not have to be precise, but they should show the basic shapes of f and S_2 on the interval $[-2,2]$.



Note that on the interval $[0,2]$, S_2 is a fairly good approximate of f. However, it is not very good on the interval $[-2,0]$, and we should not expect it to be as f is not an even function, and it is the even extension of f from $[0,2]$ to $[-2,0]$, that is used to compute the cosine series expansion of f.

3. (30) Find the solution to the following problem:

$$\begin{aligned} \Delta u &= 0, \quad 0 < x < 3,\ 0 < y < 1 \\ u(x,0) &= 0,\ u(x,1) = 1,\ 0 < x < 3, \\ \frac{\partial u(0,y)}{\partial x} &= 0,\ u(3,y) = y,\ 0 < y < 1. \end{aligned}$$

The problem is solved by breaking it into two separate problems, which lead to two function u_1 and u_2. The solution u is the sum of these two functions. The two problems that u_1 and u_2 solve are

$$\begin{aligned} \Delta u_1 &= 0, \quad 0 < x < 3,\ 0 < y < 1 \\ u_1(x,0) &= 0,\ u_1(x,1) = 0,\ 0 < x < 3, \\ \frac{\partial u_1(0,y)}{\partial x} &= 0,\ u_1(3,y) = y,\ 0 < y < 1. \end{aligned}$$

$$\begin{aligned} \Delta u_2 &= 0, \quad 0 < x < 3,\ 0 < y < 1 \\ u_2(x,0) &= 0,\ u_2(x,1) = 1,\ 0 < x < 3, \\ \frac{\partial u_2(0,y)}{\partial x} &= 0,\ u_2(3,y) = 0,\ 0 < y < 1. \end{aligned}$$

Using separation of variables to solve the first problem, $u_1\ x,y \quad g\ x\ h\ y$, we are lead to

$$\frac{g}{g} \qquad \frac{h}{h} \qquad {}^{2} \text{ or}$$

$$g\ x \quad ae^{\ x} \quad be^{\ x}$$

$$h\ y \quad c\sin\ y \quad d\cos\ y$$

These are the solutions obtained if 0. If 0, then $h\ y \quad ay \quad b$, and the boundary conditions on h force $h\ y \quad 0$ for each y. Thus, 0, and the boundary conditions on h give us $h\ y \quad \sin k\ \ y$, for k 1, 2, . The condition $\frac{u_1}{x}\ 0,y$, enables us to conclude that $g\ x \quad \cosh\ k\ \ x$. Thus,

$$u_1\ x,y \qquad \underset{k\ 1}{} b_k \cosh\ k\ \ x\ \sin\ k\ \ y \text{ , with}$$

$$y \quad u_1\ 3,y \qquad \underset{k\ 1}{} b_k \cosh\ 3k \quad \sin\ k\ \ y$$

$$b_k \cosh\ 3k \qquad 2 \ {}_{0}^{1}\ y \sin\ k\ \ y\ \ dy$$

Using the separation of variables technique on the second problem we are lead to

$$\frac{g}{g} \qquad \frac{h}{h} \qquad {}^{2} \text{ or}$$

$$g\ x \quad a\sin\ \ x \quad b\cos\ \ x$$

$$h\ y \quad ce^{\ y} \quad de^{\ y}$$

if 0. If 0, then $g\ x \quad ax \quad b$. $g\ 0 \quad 0$ implies a 0, and then $g\ 3 \quad 0$ implies b 0, or that $g\ x \quad 0$ for all x. Hence 0, and the boundary conditions that g must satisfy force

$$g\ x \quad \cos\left(\ 2k \quad 1\ \frac{}{6}x\right) \text{ for } k \quad 1, 2, \quad .$$

The condition that $h\ 0 \quad 0$ enables us to conclude that $h\ y \quad \sinh\left(\ 2k \quad 1\ \frac{}{6}y\right)$. Thus, we have

$$u_2\ x,y \qquad \underset{k\ 1}{} a_k \sinh\left(\ 2k \quad 1\ \frac{}{6}y\right)\cos\left(\ 2k \quad 1\ \frac{}{6}x\right)\text{, with}$$

$$1 \quad u_2\ x,1 \qquad \underset{k\ 1}{} a_k \sinh\left(\ 2k \quad 1\ \frac{}{6}\right)\cos\left(\ 2k \quad 1\ \frac{}{6}x\right) \text{ with}$$

$$a_k \sinh\left(\ 2k \quad 1\ \frac{}{6}\right) \quad \frac{2}{3}\ {}_{0}^{3} \cos\left(\ 2k \quad 1\ \frac{}{6}x\right) dx .$$