1. (45) Consider the following problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \pi, \quad 0 < t \\ u(0,t) &= 0, \quad u_x(\pi, t) = 0, \quad 0 < t \\ u(x,0) &= \sin x, \quad u_t(x,0) = e^x, \quad 0 < x < \pi \end{aligned}$$

Let ASTP refer to the Associated Sturm-Liouville Problem that arises from this problem when the separation of variables technique is applied.

(a) What is the ASTP ?

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \phi'(\pi) &= 0 \end{aligned}$$

(b) Derive the Rayleigh quotient expression for the eigenvalues of the ASTP. What can you say about the eigenvalues of ASTP?

Multiply the differential equation that λ and ϕ satisfy by ϕ and then integrate by parts

$$\begin{aligned} 0 &= \lambda \int_0^\pi \phi^2\, dx + \int_0^\pi \phi''\phi\, dx \\ & \lambda \int_0^\pi \phi^2\, dx + \left\{ \phi'\phi|_0^\pi - \int_0^\pi \phi'\phi'\, dx \right\} \\ &= \lambda \int_0^\pi \phi^2\, dx - \int_0^\pi (\phi')^2\, dx \end{aligned}$$

The boundary conditions, which ϕ satisfies enable us to conclude that $\phi'\phi|_0^\pi = 0$. Solving for λ we have

$$\lambda = \frac{\int_0^\pi (\phi')^2}{\int_0^\pi \phi^2\, dx}$$

This equality enables us to conclude that any eigenvalue must satisfy $\lambda \geq 0$. Moreover if 0 is an eigenvalue, then the corresponding eigenfunction must equal $\phi = ax + b$ for some constants a and b. The boundary conditions then force a and b to both equal 0. Thus, there are no non-trivial solutions to the Sturm-Liouville problem when $\lambda = 0$, and we may conclude that 0 is not an eigenvalue. Thus, every eigenvalue must be positive.

(c) What are the eigenvalues and eigenfunctions of the ASTP ?

Since the eigenvalues are positive we know that any solution of $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ is of the form $\phi = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$. The boundary condition $\phi(0) = 0$ forces $a = 0$. Thus, $\phi = \sin\sqrt{\lambda}x$, and $\phi'(\pi) = 0$ only if

$$\begin{aligned} \sqrt{\lambda}\pi &= (2k-1)\frac{\pi}{2} \text{ for } k = 1,\ 2,\ \cdots \\ \sqrt{\lambda} &= \frac{2k-1}{2} \text{ and } \lambda = \left(\frac{2k-1}{2}\right)^2 \end{aligned}$$

Thus, the eigenvalues are $\lambda = \left(\frac{2k-1}{2}\right)^2$ and the corresponding eigenfunctions are $\sin\frac{2k-1}{2}x$.

(d) How would you solve the original problem?

When separating variables with $u(x,t) = g(x)h(t)$ the function h is found to satisfy the differential equation $h'' + \lambda h = 0$. Thus,

$$h(t) = a_k\cos\left(\frac{2k-1}{2}t\right) + b_k\sin\left(\frac{2k-1}{2}t\right)$$

We look for a solution of the original problem in the form

$$u(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{2k-1}{2}t\right) + b_k\sin\left(\frac{2k-1}{2}t\right)\right)\sin\left((2k-1)\frac{\pi}{2L}x\right)$$

The constants a_k and b_k are determined from the initial conditions.

$$\begin{aligned} \sin x &= u(x,0) = \sum_{k=1}^{\infty} a_k\sin\left(\frac{2k-1}{2}x\right) \\ e^x &= u_t(x,0) = \sum_{k=1}^{\infty}\frac{2k-1}{2}b_k\sin\left(\frac{2k-1}{2}x\right) \end{aligned}$$

Thus, we must have

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi}\sin x\sin\left(\frac{2k-1}{2}x\right)\,dx \\ b_k &= \frac{4}{(2k-1)\pi}\int_0^{\pi}e^x\sin\left((2k-1)\frac{\pi}{2L}x\right)\,dx \end{aligned}$$

2. (45) Solve the following problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{x}{\pi}(\cos t + \sin t) \quad 0 < x < \pi, \quad 0 < t \\
u(0,t) &= \cos t, \quad u(\pi,t) = 1 + \sin t \quad 0 < t \\
u(x,0) &= 1, \quad 0 < x < \pi
\end{aligned}
$$

The first step is to reduce this problem to one with homgeneous boundary conditions. Set $v(x,t) = u(x,t) - r(x,t)$, where $r(x,t) = \cos t + \frac{x}{\pi}(1 + \sin t - \cos t)$. Then v satisfies the following

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \frac{x}{\pi}(\cos t + \sin t) - \left(-\sin t + \frac{x}{\pi}(\cos t + \sin t)\right) \quad \text{or} \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \sin t, \quad 0 < x < \pi, \quad 0 < t \\
v(0,t) &= 0, \quad v(\pi,t) = 0, \quad 0 < t \\
v(x,0) &= 0, \quad 0 < x < \pi
\end{aligned}
$$

The eigenvalues and eigenfunctions for the Sturm-Liouville problem, which arises from solving the associated homogeneous partial differential equation and boundary conditions are $\lambda = n^2$, $\phi_n(x) = \sin nx$, for $n = 1, 2, \cdots$. So we look for the solution v in the form

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nx$$

Moreover, since the initial condition $v(x,0) = 0$, we must have $a_n(0) = 0$ for each n. Placing this expression into the differential equation for v we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} a_n'(t) \sin nx &= \sum_{n=1}^{\infty} \left(-n^2\right) a_n(t) \sin nx + \sin t \\
\sum_{n=1}^{\infty} \left(a_n'(t) + n^2 a_n(t)\right) \sin nx &= \sin t = \sum_{n=1}^{\infty} b_n(t) \sin nx \quad \text{or} \\
a_n'(t) + n^2 a_n(t) &= b_n(t), \; a_n(0) = 0, \quad \text{with} \\
b_n(t) &= \frac{2\sin t}{\pi} \int_0^{\pi} \sin nx \, dx, \quad \text{for } n = 1, 2, \cdots \\
&= \frac{2\sin t}{\pi} \left(\frac{1 - \cos(n\pi)}{n}\right) \\
&= \begin{cases} 0 & n \text{ even} \\ \frac{4\sin t}{n\pi} & n \text{ odd} \end{cases}
\end{aligned}
$$

The solution of this initial value prolem is

$$a_n(t) = \frac{4e^{-n^2 t}}{n^5\pi + n\pi} + \frac{4(n^2 \sin t - \cos t)}{n\pi(n^4 + 1)}$$

when n is odd. Thus, we finally have

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \left(\frac{4e^{-(2n-1)^2 t}}{(2n-1)^5 \pi + (2n-1)\pi} + \frac{4((2n-1)^2 \sin t - \cos t)}{(2n-1)\pi\left((2n-1)^4 + 1\right)} \right) \sin(2n-1)x \\
&\quad + \cos t + \frac{x}{\pi}(1 + \sin t - \cos t)
\end{aligned}
$$

3. (10) Consider the Sturm-Liouville problem

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi &= 0, \quad 0 < x < L \\ \phi(0) &= 0 \\ \phi'(L) &= 0 \end{aligned}$$

where p, q, and σ are continuous on the interval $[0, L]$, and both p and σ are positive and bounded away from zero. Let $\mathrm{L}(\phi) = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$

The following differential equality may be useful

$$\mathrm{L}(u)v - u\mathrm{L}(v) = \frac{d}{dx}\left(pu'v - puv'\right).$$

(a) Use the differential inequality to show that $\int_0^L \mathrm{L}(u)\, v\, dx = \int_0^L u\mathrm{L}(v)\, dx$ for every pair of functions u and v which satisfy the boundary conditions of the Sturm-Liouville problem.

$$\begin{aligned} \int_0^L \left(\mathrm{L}(u)v - u\mathrm{L}(v)\right) dx &= \left(pu'v - puv'\right)\Big|_0^L \\ &= p(L)u'(L)v(L) - p(L)u(L)v'(L) \\ &\quad - \left(p(0)u'(0)v(0) - p(0)u(0)v'(0)\right)\Big| \\ &= 0 \end{aligned}$$

Thus, we have $\int_0^L \mathrm{L}(u)\, v\, dx = \int_0^L u\mathrm{L}(v)\, dx$.

(b) Show that if λ_1 and λ_2 are two different eigenvalues of the Sturm-Liouville problem with corresponding eigenfunctions ϕ_1 and ϕ_2, then

$$\int_0^L \phi_1(x)\phi_2(x)\sigma(x)\, dx = 0.$$

$$\begin{aligned} \lambda_1 \int_0^L \phi_1(x)\phi_2(x)\sigma(x)\, dx &= \int_0^L \lambda_1\sigma(x)\phi_1(x)\phi_2(x)\, dx \\ &= -\int_0^L \mathrm{L}(\phi_1)\phi_2\, dx \\ &= -\int_0^L \phi_1\mathrm{L}(\phi_2)\, dx \\ &= \int_0^L \phi_1(x)\lambda_2\phi_2(x)\sigma(x)\, dx \\ &= \lambda_2 \int_0^L \phi_1(x)\phi_2(x)\sigma(x)\, dx \end{aligned}$$

This implies that

$$(\lambda_1 - \lambda_2)\int_0^L \phi_1(x)\phi_2(x)\sigma(x)\, dx = 0$$

Since $\lambda_1 - \lambda_2 \neq 0$, we must have $\int_0^L \phi_1(x)\phi_2(x)\sigma(x)\, dx = 0$.