

1. (15) The formula $V(s, t) = 10e^{-r(T-t)}N(d_2)$ gives the price of a cash or nothing option, which expires at time T , where the strike price is \$40, and

$$d_2 = \frac{\ln(s/40) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

- (a) What is the cash payment?

To determine the cash payment at expiration we need to take the limit of $V(s, t)$ as t approaches T from below. To do this we first note that

$$\lim_{t \rightarrow T^-} d_1(s, t) = \begin{cases} \infty & \text{if } s > 40 \\ -\infty & \text{if } s < 40 \end{cases}$$

Thus, we have

$$\lim_{t \rightarrow T^-} V(s, t) = \begin{cases} 10 & \text{if } s > 40 \\ 0 & \text{if } s < 40 \end{cases}$$

- (b) What has to happen at expiration for the payment to be made?

The price of the stock has to be more than \$40.

2. (15) A broker is explaining a new type of option on a particular stock, and claims it is the best way to safeguard your portfolio. The broker even shares with you the mathematical formula for determining the price of this option. It is

$$V(S, t) = Se^{-(T-t)^2/2} - Xe^{-r(T-t)},$$

where S is the price of the stock, T is the time of expiration, t is time, and r is the short interest rate. Does this formula give the price of an option?

V will be the price of an option if it satisfies the Black-Scholes partial differential equation. The various partial derivatives of V are

$$\frac{\partial V}{\partial S} = e^{-(T-t)^2/2}, \quad \frac{\partial^2 V}{\partial S^2} = 0$$

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial}{\partial t} \left(Se^{-(T-t)^2/2} - Xe^{-r(T-t)} \right) \\ &= Se^{-\frac{1}{2}(T-t)^2} T - Se^{-\frac{1}{2}(T-t)^2} t - Xre^{-r(T-t)} \end{aligned}$$

Putting these terms into the Black-Scholes partial differential equation we get

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &= Se^{-\frac{1}{2}(T-t)^2} T - Se^{-\frac{1}{2}(T-t)^2} t - Xre^{-r(T-t)} \\ &+ rSe^{-(T-t)^2/2} - r \left(Se^{-(T-t)^2/2} - Xe^{-r(T-t)} \right) \\ &= Se^{-\frac{1}{2}(T-t)^2} T - Se^{-\frac{1}{2}(T-t)^2} t \neq 0 \end{aligned}$$

Since $V(S, t)$ does not satisfy the Black-Scholes partial differential equation for all values of S and t , it does not equal the price of an option.

3. (20) You want to purchase a European style option, which pays one share of stock at expiration if the price of the stock is between \$50 and \$60 and nothing otherwise. Assume the current price of the stock is \$52, the stock's annualized volatility is $\sigma = 0.4$, and that the short interest rate $r = 0.05$. How much should you pay for this option which expires in 3 months?

The price of a stock or nothing option is given by $V(S, t) = SN(d_X)$, where

$$d_X = \frac{\ln(S/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

and τ is the time to expiration. Since we want to get a share of stock if the price is more than 50 and less than 60, we are essentially buying one of these options with a strike price of 50 and selling one with a strike price of 60. The values of these two options are

$$\begin{aligned} d_{50} &= \frac{\ln(52/50) + (0.05 + 0.4^2/2)(1/4)}{0.4\sqrt{1/4}} \approx 0.3586 \\ d_{60} &= \frac{\ln(52/60) + (0.05 + 0.4^2/2)(1/4)}{0.4\sqrt{1/4}} \approx -0.553 \end{aligned}$$

Thus the price of the combined option is

$$\begin{aligned} V_{50}(52, 1/4) - V_{60}(52, 1/4) &\approx 52N(d_{50}) - 52N(d_{60}) \\ &\approx 33.283 - 15.087 \\ &= 18.196 \end{aligned}$$

4. (40) A broker short sells 1000 shares of a stock for \$50 a share. The stock has an annualized volatility of 0.35. Assume a short interest rate of $r = 0.04$. The broker decides to hedge his position by buying call options, with a strike price of \$55 and expiration in 4 months. His position can then be described by

$$1000(aV - S),$$

where V is the price of the option and S is the price of the stock.

- (a) Suppose the broker decides to buy one 1/2 call option for each share of stock shorted. That is, the broker buys 500 call options. Plot the broker's profit divided by 1000 at expiration as a function of the stock price S , for $0 \leq S \leq 100$.

Since the broker sold a share at \$50 and bought 1/2 of a call option, the initial cash position is,

$$50 - \frac{2.40}{2} = 48.80,$$

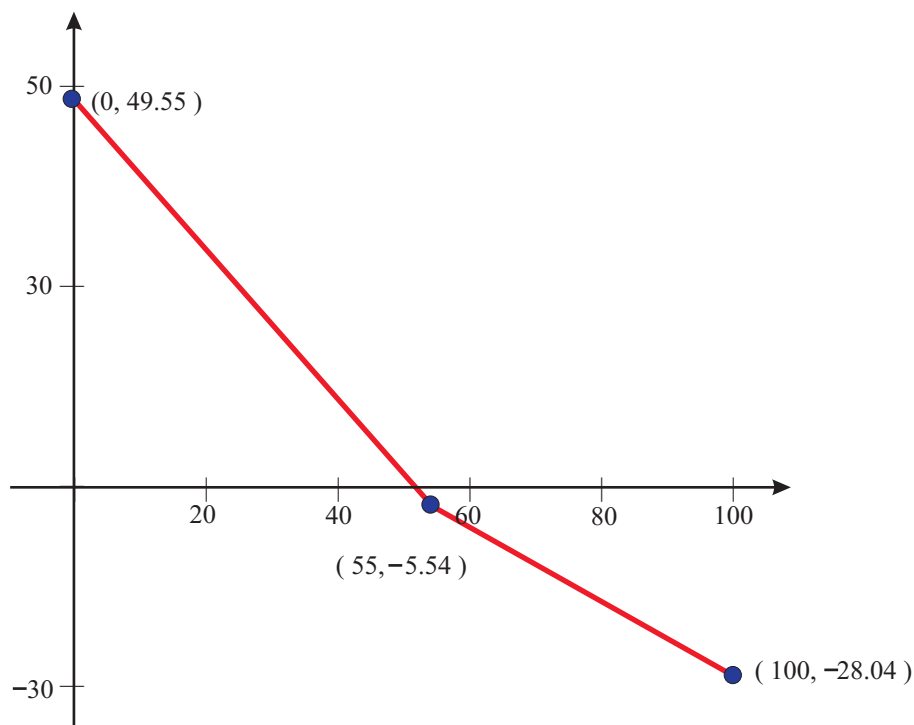
where 2.40 is the cost of one call option. The broker's position is then $-S + \frac{V}{2}$, since the share of stock and the 1/2 option need to be bought and sold respectively. At expiration this is equal to

$$-S + \frac{(S - 55)^+}{2}.$$

Thus, at expiration, the profit will equal

$$48.80e^{0.04/3} - S + \frac{(S - 55)^+}{2}.$$

A plot is shown below.



- (b) What must a equal if the broker δ hedges his position. There are two answers to this question one which is generic in terms of V and the second a number for this particular case. You are to supply both answers.

The broker's position is $\Pi = aV - S$. Since we want the derivative of Π with respect to S to equal zero, we have

$$\frac{\partial \Pi}{\partial S} = a \frac{\partial V}{\partial S} - 1 = 0.$$

Thus,

$$a = \frac{1}{\partial V / \partial S}.$$

Since $\frac{\partial V}{\partial S} = N(d_1)$ when V is the price of a European call option, we have

$$a \approx \frac{1}{.3803} \approx 2.629.$$

- (c) What amount of cash must the broker borrow or have to invest after setting up the δ hedged portfolio?

The broker short sold 1000 shares and bought 2,629 options. Thus, the amount of cash is

$$1000(50) - 2629(2.40) = 43,690.00$$

- (d) Assume that at three months to expiration the stock price falls to \$45 per share. What should the broker do to δ hedge the position? What is the new cash position?

The hedge value is now 6.323. Don't forget, when computing $\frac{\partial V}{\partial S}$, the time to expiration is now 3 months with a stock price of \$45. Since the value of δ increased, this means that the broker has to buy some more options. The cash needed to buy these options is

$$1000(0.605)(6.323 - 2.629) = 2,234.90$$

Thus, the new cash position is

$$43,690.00e^{0.04/12} - 2,234.90 = 41,601.00$$

5. (10) In deriving both the Black-Scholes formula for the price of a European call option and the Black-Scholes partial differential equation the term

$$(dS)^2 = (\mu S dt + \sigma S dB)^2$$

is replaced with the term $\sigma^2 S^2 dt$. Explain/justify this approximation.

The crucial fact used in this approximation is that $dB = Z\sqrt{dt}$, where Z is the standard normal random variable. Thus, we have

$$\begin{aligned}(dS)^2 &= (\mu S dt + \sigma S dB)^2 = (\mu S dt + \sigma S Z\sqrt{dt})^2 \\ &= \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 Z (dt)^{3/2} + \sigma^2 S^2 Z^2 dt \\ &\approx \sigma^2 S^2 Z^2 dt\end{aligned}$$

Only the last summand is kept as the first two contain powers of dt higher than 1. The next step is to note that the expected value of Z^2 is one, and to replace Z^2 with 1. Thus, we have

$$(dS)^2 \approx \sigma^2 S^2 dt$$