

1. (20) Let  $C$  denote the price of an American call option on a share of non-dividend paying stock. Let  $X$  denote the strike price,  $e^r$  the annual interest, and  $\tau$  the time to expiration.

(a) Justify the inequality:

$$C \geq S - e^{-r\tau} X.$$

Suppose that  $C < S - e^{-r\tau} X$ , then we have  $C + e^{-r\tau} X < S$ . Short sell one share of stock for  $S$  dollars, and buy one option for  $C$  dollars invest the rest, namely  $S - C$ .

At maturity exercise the option which costs  $X$  dollars and use the stock purchased to close out the short stock position. The amount of money left is

$$(S - C)e^{r\tau} - X > (e^{-r\tau} X)e^{r\tau} - X > 0.$$

Thus, if the given inequality is not true there is an arbitrage opportunity. Moreover, if the price of the stock is less than the strike price an even greater profit will be made.

- (b) Explain why this inequality implies it is never wise to exercise an American call option before it matures.

Set  $IV(C) = C - S + e^{-r\tau} X \geq 0$ . Then we have

$$\begin{aligned} C &= S - e^{-r\tau} X + IV(C) \Rightarrow \\ C - (S - X) &= (X - e^{-r\tau} X) + IV(C) \end{aligned}$$

Thus, the value of an American call option is never less than its expiration value. So it makes no financial sense to exercise the option early. If one wants to close out the option position, don't exercise it, sell it.

2. (25) The Geometric Brownian Motion model for stock prices is  $S_t = S_0 e^{\sigma B_t} e^{(\mu - \sigma^2/2)t}$ .

(a) Show that  $S_t$  satisfies the differential equation  $dS = \mu S dt + \sigma S dB$ .

Using Taylor's expansion we have to first order terms in  $dt$

$$\begin{aligned}
 dS &= \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial B} dB + \frac{1}{2} \frac{\partial^2 S}{\partial t^2} (dt)^2 + \frac{\partial^2 S}{\partial t \partial B} dt dB + \frac{1}{2} \frac{\partial^2 S}{\partial B^2} (dB)^2 + \dots \\
 &= \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial B} dB + \frac{1}{2} \frac{\partial^2 S}{\partial B^2} (dB)^2 \\
 &= \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial B} dB + \frac{1}{2} \frac{\partial^2 S}{\partial B^2} dt \\
 &= \left( \frac{\partial S}{\partial t} + \frac{1}{2} \frac{\partial^2 S}{\partial B^2} \right) dt + \frac{\partial S}{\partial B} dB \\
 &= \left( (\mu - \sigma^2/2) S + \frac{1}{2} \sigma^2 S \right) dt + (\sigma S) dB \\
 &= \mu S dt + \sigma S dB
 \end{aligned}$$

(b) Explain why  $\mu$  is set equal to  $r$  in deriving the Black-Scholes formula for the price of a European call option.

When the drift term  $\mu$  is set equal to  $r$  in the Geometric Brownian Motion model of stock prices we have the following relationship between the expected value of  $S_t$  and  $S_0$ , namely

$$e^{r\tau} S_0 = E[S_\tau],$$

where  $E[S_t]$  denotes the expected value of  $S_t$ . The point of this is to then construct a portfolio of the form

$$\Pi_0 = aS_0 + b \text{ and } \Pi_t = aS_t + be^{rt},$$

where  $b$  is an amount of cash, which is invested, and  $a$  and  $b$  are such that the value of the portfolio is the same as the value of an option, typically a call option. In any case we have the following:

$$\begin{aligned}
 \Pi_t &= aS_t + be^{rt} = \Pi_t \\
 &= aS_t + (\Pi_0 - aS_0) e^{rt} \Rightarrow \\
 \Pi_t - e^{rt}\Pi_0 &= a(S_t - S_0 e^{rt}).
 \end{aligned}$$

Thus, we must have

$$e^{r\tau}\Pi_0 = E[\Pi_\tau].$$

The next step is to compute the expected value of the option at expiration, and then use the above equation to determine the current value of the option.

3. (20) Derive the Black-Scholes partial differential equation. That is, show that the price  $V$  of an option on a non-dividend paying stock satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

A portfolio  $\Pi = aS - V$  is constructed so that  $d\Pi$  is deterministic.

$$\begin{aligned} d\Pi &= a dS - dV \\ &= a dS - \left( \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS \right) \\ &= \left( a - \frac{\partial V}{\partial S} \right) dS - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt \end{aligned}$$

The term  $\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}$  arises from the fact that  $(dS)^2 = \sigma^2 S^2 (dB)^2 = \sigma^2 S^2 dt$ . If  $a = \frac{\partial V}{\partial S}$ , then we have

$$d\Pi = - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt.$$

Since this is deterministic, i.e., risk free, we must have  $d\Pi = r\Pi dt$ . This implies

$$r(aS - V) = - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right).$$

Hence,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + raS - rV &= 0 \\ \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &= 0 \end{aligned}$$

4. (35) The price of a share of X-Corp stock is currently \$55. Assume that the yearly interest  $e^r = 1.02$ , and that the stock's volatility is 0.4.

(a) Determine the price of a European call option with a strike price of \$55, and expiration in three months.

Using the Black-Scholes formula for the price of a European call option we have

$$\begin{aligned} C &= 55N(d_1) - \frac{55}{(1.02)^{1/4}}N(d_2) \\ &= 55N(0.1247) - \frac{55}{1.02^{1/4}}N(-0.0753) \\ &= 55(0.5496) - \frac{55}{1.02^{1/4}}(0.47) \\ &= 55(0.5496) - \frac{55}{1.02^{1/4}}(0.47) \\ &= 4.506 \end{aligned}$$

(b) Determine the price of a European put option with a strike price of \$55, and expiration in three months.

The easiest way to compute the price of this put option is to use the put-call parity relation. Thus,

$$\begin{aligned} P &= C + Xe^{-r\tau} - S \\ &= 4.507 + \frac{55}{1.02^{1/4}} - 55 \\ &= 4.236 \end{aligned}$$

(c) Suppose you set up a protective put. That is, you buy one share of stock and one put option. Assume that at three months time the stock price has fallen to \$54.50. What is the value of this portfolio at expiration? Have you made or lost money?

The value of the portfolio is  $54.50 + (55 - 54.50) = 55$ . However, the initial cost of the portfolio was  $55 + 4.236 = 59.236$ . After three months the amount paid back is  $59.236(1.02)^{1/4} = 59.53$ . Thus, a net loss of 4.53 was had.