Math 425 **Exam 3** April 25, 2002

1. (30) Assume a stock pays a continuous dividend at a yield of D_y. That is, if during the time interval dt the stock has value S, then the amount of the dividend is $D_y S\,dt$.

a. Let $V\;S,t$ denote the value of an option on this stock with strike price X. Show that

$$dV \quad \left(\frac{V}{t} \quad \frac{1}{2} \; {}^2S^2 \frac{{}^2V}{S^2} \right) dt \quad \frac{V}{S}\, dS.$$

b. Let $V\;S,t$ denote the value of an option on this stock with strike price X. Show that V satisfies the partial differential equation

$$\frac{V}{t} \quad \frac{1}{2} \; {}^2S^2 \frac{{}^2V}{s^2} \qquad r \quad D_y \; S\frac{V}{S} \quad rV \quad 0,$$

where r denotes the short interest rate. That is e^r is the future value of one dollar in one year.

c. Let $C\;S,t$ denote the price of a European call option on a non-dividend paying stock with strike price X, and it is assumed that the interest rate for this option is $r \quad D_y$. Show that $C_D\;S,t \qquad e^{\;Dy\;T\;t}\;C\;S,t$, where T is the time of expiration, satisfies the partial differential equation in part a.

d. Suppose that $S \quad 18$, $T \quad 3$ months, $r \quad 0.01$, $D_y \quad 0.01$, and that $\quad 0.4$. Determine the value of a European call option on this stock.

2. (25) Let a non-dividend paying stock have the following properties $\quad 0.3$, $S_0 \quad 25$, and suppose the short rate is $r \quad 0.02$. That is, the future value of 1 dollar in one year is $e^{0.02}$. Suppose $P\;S,t$ denotes the price of a European put option, with strike price $X \quad 24$.

a. What is the price of this option?

b. Suppose a broker sells 1000 of these put options. How many shares of stock should be bought or sold to -hedge this sale?

c. One week later the stock price has fallen to \$24. What should the broker do to rehedge the position?

3. (20) Suppose $V\;S,t$ denotes the value of an option, which at expiration has the payoff

$$V\;S,T \qquad \begin{cases} 0, & S \quad X \\ 10, & X \quad S \quad X \;\; 2 \\ 15, & X \;\; 2 \quad S \quad X \;\; 5 \\ 0, & X \;\; 5 \quad S \end{cases}$$

Explain how you would calculate the value of this option at $t \quad 0$.

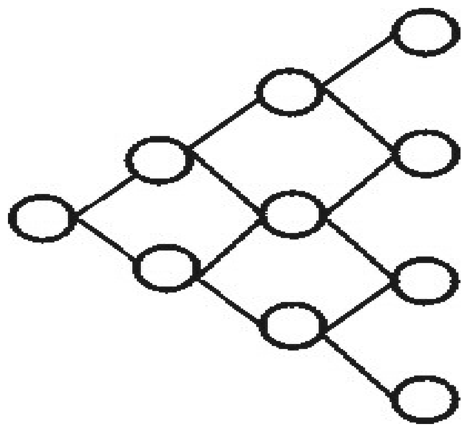
4. (25) If a binomial tree is constructed to approximate the Geometric Brownian Motion model of stock prices, we have $u \quad e^{\sqrt{\ t}}$ and $d \quad e^{\sqrt{\ t}}$. Where as usual $S_u \quad uS_0$ and $S_d \quad dS_0$.

a. The value of q, the probability of an up move by the stock price, which is used in this binomial tree model is

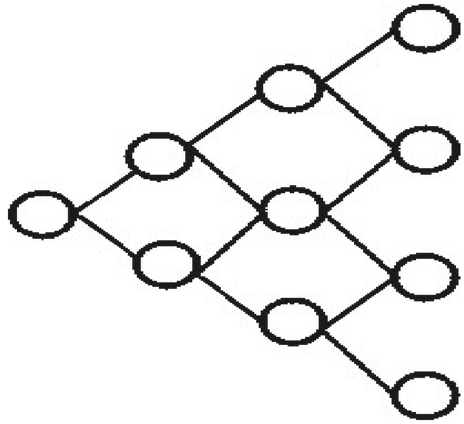
$$q \quad \frac{1}{2} \quad \sqrt{\ t}\,\frac{r \qquad {}^2/2}{2}.$$

This number is not the arbitrage free probability, q, that one associates with a binary tree. What does q equal, and how did we get from q to q. You don't have to do any calculations for this one, just explain what happened.

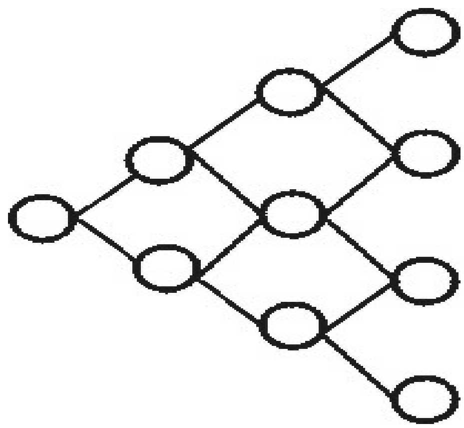
b. Suppose $\quad 0.4$, $T \quad 1/12$, and $r \quad 0.01$. Fill in the values of the following stock price tree if $S_0 \quad 15$:



c. Suppose *V* is the value of a European put option with *X* 15. Fill in the following option tree.



d. Suppose *V* is the value of an American put option with *X* 15. Fill in the following option tree.



The following formulas may be helpful

Black-Scholes formula for the price of a European call option on a non-dividend paying asset:

$$C \ S,t \quad SN \ d_1 \quad e^{\ r \ T \ t} XN \ d_2$$

$$d_1 \quad \frac{\ln \ S/X \quad r \quad {}^2/2 \ \ T \quad t}{\sqrt{T \quad t}}$$

$$d_2 \quad d_1 \quad \sqrt{T \quad t}$$

$$N \ x \quad \frac{1}{\sqrt{2}} \ {}^{x} \ e^{\ z^2/2} dz$$

$$\frac{C}{S} \quad N \ d_1$$

Black-Scholes partial differential equation for an option on a non-dividend paying asset:

$$\frac{V}{t} \quad \frac{1}{2} \ {}^2S^2 \frac{{}^2V}{s^2} \quad rS\frac{V}{S} \quad rV \quad 0$$

Let $V \ S,t$ denote the price of an option. Then we know that

$$V \ S,t \quad Xe^{\ \ln \ S/X} e^{\ \ T \ t \ \ {}^2/2} u\left(\ln \ S/X \ , \frac{{}^2}{2} \ T \quad t \ \right),$$

where the function $u \ x,t$ satisfies the heat equation, T is the time to expiration of the option, X is the strike price, and

$$\frac{{}^2 \quad 2r}{2 \ {}^2}, \text{ and } \quad \left(\frac{{}^2 \quad 2r}{2 \ {}^2}\right)^2.$$

Solution to heat equation:

If $u \ x.t$ satisfies the heat equation $\frac{u}{t} \quad \frac{{}^2u}{x^2}$, for x and $t \quad 0$, and $u \ x,0 \quad x$ for x , then

$$u \ x,t \quad \frac{1}{\sqrt{4 \ t}} \quad e^{\ x \quad {}^2/4t} d \ .$$

Put/call parity formula for European options:

$$S \quad P \quad C \quad e^{\ r} X$$