1. (30) Let F represent the forward price of a commodity at time n. Suppose $100r\%$ is the interest rate per time period. Let S_0 represent the price of the commodity right now. Show, by use of the no arbitrage principle, that F must equal

$$F = S_0(1+r)^n \, .$$

Let's assume that the forward price F is greater than $S_0(1+r)^n$. So enter into a forward contract that obligates you to sell the commodity at the price F. Borrow S_0 and buy the commodity. At time $t = 1$ honor the forward contract by selling the commodity for F dollars, use this to pay off your loan, which leaves a profit of $F - S_0(1+r)^n > 0$ dollars.

If F is less than $S_0(1+r)^n$, then enter into a forward contract where you agree to buy the commodity for F dollars. Short sell the commodity for S_0 dollars and invest the money. At time $t = 1$ take F dollars from your investment, honor the forward contract by purchasing the commodity and then close out your short position. This leaves a profit of $S_0(1+r)^n - F$ dollars.

2. (50) The table below represents the possible prices of a share of stock

$t = 0$	$t = 1$	$t = 2$
		110
	105	
		103
100		
		98
	90	
		85

Number the nodes 1(100), 2(105), 3(90), and assume that $r = 0.02$.

a. Compute the risk neutral probability p_2 at node 2.

For node 2 the values of u_2, d_2, and p_2 are

$$u_2 = \frac{110-105}{105} = \frac{1}{21} = 0.04762$$

$$d_2 = \frac{103-105}{105} = -\frac{2}{105} = -0.01905$$

$$p_2 = \frac{0.02 - \left(-\frac{2}{105}\right)}{\frac{1}{21} - \left(-\frac{2}{105}\right)} = 0.58571$$

b. What is the conditional expected value of the stock price at time $t = 2$ with respect to the risk neutral probability given that $S(1) = 90$? (Think)

The expected value of a stock price using the risk neutral probability satisfies the equation

$$\begin{aligned} E[S(2)|S(1) = 90] &= 90(1.02) \\ &= 91.8 \end{aligned}$$

c. What is the risk neutral probability that $S(2) = 103$?

We first need to calculate p_1. The values of u_1, d_1, and p_1 are

$$u_1 = \frac{105 - 100}{100} = 0.05$$
$$d_1 = \frac{90 - 100}{100} = -0.1$$
$$p_1 = \frac{0.02 - (-0.1)}{0.05 - (-0.1)} = 0.8$$

The probability that $S(2) = 103$ equals

$$\begin{aligned}\Pr(S(2) = 103) &= \Pr(S(1) = 105)\Pr(S(2) = 103|S(1) = 105) \\ &= p_1(1 - p_2) = (0.8)(1 - 0.58571) \\ &= 0.33143\end{aligned}$$

3. (50) Suppose that we have the following values for a binary tree model of stock prices:

$$u = 0.03, d = -0.01, r = 0.02, p = 0.75$$

Suppose $S(0) = 1$. Let $A^P(t)$ denote the value of an American put option at time t, with strike price $X = 1$, and let $A^C(t)$ denote the price at time t of an American call option with strike price $X = 1$. Assume both options mature after one time tick.

a. $p = \frac{r - d}{u - d}$ is the risk neutral probability. Explain why, p is risk neutral.

If we compute the expected value of the stock price at $t = 1$ we have

$$\begin{aligned}E[S(1)] &= pS_u + (1 - p)S_d \\ &= \frac{r - d}{u - d}(1 + u)S_0 + \left(\frac{u - r}{u - d}\right)(1 + d)S_0 \\ &= \left(\frac{r - d}{u - d}(1 + u) + \left(\frac{u - r}{u - d}\right)(1 + d)\right)S_0 \\ &= (1 + r)S_0 .\end{aligned}$$

That is, the expected value is exactly equal to the value a risk free investment would have after 1 time tick.

b. Determine $A^C(0)$.

The possible values of the call at $t = 1$ are 0.03 and 0. The present value of the expected value of these prices is the value of the call at $t = 0$ and equals

$$A^C(0) = \frac{0.75 * 0.03 - 0}{1.02} = 0.022059$$

c. Determine $A^P(0)$.

The value of the call is the larger of the present value of the expected value at $t = 1$ or what the call is worth if cashed in at time $t = 0$. Since this later value is zero, we only need to compute the former value, which equals

$$A^P(0) = \frac{0 + (1 - 0.75) * 0.01}{1.02} = 0.0024510$$

4. (20) The following two tables show the binomial trees for a stock and a European Put with strike price $X = 100$, $r = 0.03$(this is for the time period of the tree), and $A(0) = 100$.

	105		0
Stock: 100		Put: 0.5548	
	98		2

A broker decides to offer this put option to his customers. He will sell it for 60 cents and buy it for 50 cents. He winds up buying 10,000 of these options from a customer.

a. If he does nothing, what are the possibilities he faces when the put option matures?

If the stock increases in value then the put options he has purchased expire worthless, so he has a loss of

$$\text{loss} = 10000(0.50) = 5000 \text{ dollars.}$$

If the stock decreases in value then the put options are worth $2 * 10000 = 20,000$ dollars, so the broker would net $\$15,000$.

This answer does not take into account that the broker's cost is not only the $5000 he paid for the options, but also the lost income if this money had been invested at 3%.

b. The broker decides to hedge his position. What should he do, and what will his profit or loss be when the option matures?

To hedge his position he computes the hedge ratio which equals

$$\text{hedge ratio} = \frac{0-2}{105-98} = -\frac{2}{7} .$$

He then buys/sells x shares of stock, where x equals, note $z = 10000$

$$x = -z\left(\frac{-2}{7}\right) = \frac{2}{7}10000 = 2857.1428 .$$

Since x is positive he'll buy the shares. To do this he arranges to borrow enough money to that the value of his portfolio at time 0 is 0. Thus, y must equal

$$y = \frac{10000}{100}\left(\left(\frac{-2}{7}100 - 0.5548\right)\right)$$
$$= -2914.3$$

The brokers portfolio is therefore $(x,y,z) = \left(\frac{20000}{7},\ -2914.3\ ,\ 10000\right)$, and the value of this portfolio at time t equals

$$V(t) = xS(t) + yA(t) + zD(t) .$$

If round off errors are ignored, then the values of $V(0)$ and the possible values of $V(1)$ are all 0. What the broker has left is the difference between what the option is worth and what he paid for it. This amounts to 0.0548 dollars per option. This is invested at 3% so when the options expire the broker has made a net profit of

$$10000 * 0.0548 * (1.03) = 548.0 * (1.03)$$
$$= 564.44 \text{ dollars} .$$