1. (50) The table below represents the possible prices of a share of stock, and the possible values of an option whose value at $t = 1$ depends on the stock price at $t = 1$.

$$\text{Stock: } 100 \begin{matrix} 105 \\ \\ 90 \end{matrix}, \quad \text{Option: } ? \begin{matrix} 5 \\ \\ 3 \end{matrix}$$

Assume $A(0) = 50$ and $A(1) = 52$.

a. What is the general definition of the risk neutral probability? Are there any conditions that a generic binomial stock price model must satisfy in order for a risk neutral probability to exist?

A probability function p is said to be risk neutral if the expected value of the stock prices at $t = 1$ satisfies

$$(1 + r)S(0) = \mathrm{E}[S(1)] . \qquad (*)$$

Necessary and sufficient conditions for the existence of such a risk neutral probability are that the values of u and d, where

$$u = \frac{S_u - S(0)}{S(0)}, \ d = \frac{S_d - S(0)}{S(0)},$$

satisfy the inequalities

$$-1 < d < r < u .$$

When these conditions are satisfied we set $p = \frac{r-d}{u-d}$ and we have $0 < p < 1$, so p is a probability function, and equation $(*)$ is satisfied

b. What is the risk neutral probability in this example?

In this example $u = 0.05$, $d = -0.1$, and $r = 0.04$. So

$$p = \frac{0.14}{0.15} = \frac{14}{15} \approx 0.93333$$

c. Compute $\mathrm{E}[S(1)]$ and $\mathrm{E}[D(1)]$, where $D(t)$ denotes the value of the option at time t.

$$\mathrm{E}[S(1)] = (1.04)S(0) = 104$$

$$\mathrm{E}[D(1)] = \frac{14}{15}(5) + \frac{1}{15}(3) = \frac{73}{15} \approx 4.8667$$

d. What is the value of $D(0)$?

$$D(0) = \frac{\mathrm{E}[D(1)]}{1.04} \approx \frac{4.8667}{1.04} = 4.6795$$

e. If a broker buys 50,000 options what should she do to hedge her position. That is, what are the values of x, y, and z. Where as usual x is the number of shares of stock, y the number of shares of A, and z is the number of options.

$$
\begin{aligned}
z &= 50000 \\
x &= -z\frac{\Delta D}{\Delta S} = -50000\frac{2}{15} = -\frac{20\,000}{3} \approx -6666.70 \\
y &= \frac{-xS(0) - zD(0)}{A(0)} = \frac{z\frac{\Delta D}{\Delta S}S(0) - zD(0)}{A(0)} \\
&= \frac{z}{A(0)}\left(\frac{\Delta D}{\Delta S}S(0) - D(0)\right) = \frac{50000}{50}\left(\frac{2}{15}100 - 4.6795\right) \\
&\approx 8653.80\,.
\end{aligned}
$$

f. Assuming that the broker bought the options for \$4.50 each, what is the broker's profit or loss with the hedged position?

Since the broker only had to borrow \$4.50 and not \$4.6795 for each option his profit after hedging will be the value of investing this difference, which equals

$$50000(4.6795 - 4.5)(1.04) = 9334.00,$$

dollars.

2. (30) The put call parity formula relates the values of a European call and put that have the same strike price X and time to maturity T:

$$C^E(0) - P^E(0) = S(0) - Xe^{-rT}\,.$$

In class we verified this formula by use of what we know about forward contracts.

a. Explain why the portfolio consisting of $C^E - P^E$ looks like a long forward contract with forward price X.

At maturity a the holder of a long forward contract with a forward price of X dollars must buy the asset for X dollars, so the profit or loss will be

$$S(T) - X\,.$$

With a profit if $S(T) > X$ and a loss or no gain otherwise. Also at maturity the value of the option position is

$$
\begin{aligned}
C^E(T) - P^E(T) &= (S(T) - X)^+ - (X - S(T))^+ \\
&= S(T) - X\,.
\end{aligned}
$$

Since the values of these two different portfolios are exactly the same at maturity, the second one consisting of the two options does indeed look like a long forward contract.

b. Give an argument, based on the assumption that arbitrage is not possible, to show that

$$C^E(0) - P^E(0) > S(0) - Xe^{-rT}$$

cannot happen.

Suppose that does indeed happen, then do the following:

Sell the call, buy the put, and buy the stock, which means that at maturity you will owe the following amount. Note: most likely this expression is negative.

$$[C^E(0) - P^E(0) - S(0)]e^{rT}$$

There are two cases to consider: either $S(T) \geq X$ or $S(T) < X$.

If $S(T) \geq X$, then the holder of the call option will exercise it and give you X dollars, and you will turn over the stock that you bought, and the put option will expire worthless. You will then have

$$X + [C^E(0) - P^E(0) - S(0)]e^{rT} = [Xe^{-rT} + C^E(0) - P^E(0) - S(0)]e^{rT},$$

and by assumption this is a positive quantity.

If $S(T) < X$, then exercise the put option. That is, turn over the stock for X dollars; the call option will expire worthless. The value of your position is as it was above, i.e., positive. Thus, no matter what if

$$C^E(0) - P^E(0) > S(0) - Xe^{-rT}$$

there is an arbitrage opportunity.

3. (20) A European put is a monotonic function of its strike price. That is, it is either an increasing or decreasing function of the strike price. Which way is correct, and give an argument to verify your claim.

The claim is that the value of a put option is an increasing function of the strike price. That is, if $X_1 < X_2$, then

$$P^E(X_1) \leq P^E(X_2) .$$

Since a put option gives you the right to sell an asset for X dollars, the greater X is the more money you should be able to realize, thus making the value of the option greater. For an arbitrage argument suppose we have $P^E(X_1) > P^E(X_2)$. Then sell $P^E(X_1)$, and buy $P^E(X_2)$, and invest the difference which brings in $[P^E(X_1) - P^E(X_2)]e^{rT}$ dollars. At maturity if the holder of the "X_1" put option decides to exercise it you'll pay him X_1 dollars for the asset and then immediately sell the asset for X_2 dollars by exercising the second put. Thus, giving you a profit of

$$[P^E(X_1) - P^E(X_2)]e^{rT} + X_2 - X_1 > 0$$

dollars. If the option is not exercised then you have at the least a profit of

$$[P^E(X_1) - P^E(X_2)]e^{rT} > 0$$

dollars. This arbitrage opportunity means $P^E(X_1) > P^E(X_2)$ cannot happen.

The following formulas may be helpful in working the last problem:

$$\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\alpha(x-\tau)^2}\, dx = 1$$

$$\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} x\, e^{-\alpha(x-\tau)^2}\, dx = \tau$$

$$\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} x^2\, e^{-\alpha(x-\tau)^2}\, dx = \tau^2 + \frac{1}{2\alpha} .$$

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4. (50) Let Λ denote a random variable for which

$$\Pr(\Lambda \le a) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{a} e^{-\sigma(\lambda-1)^2}\, d\lambda \,.$$

Suppose we decide to model stock prices with the following random variable

$$S(t) = S(0)e^{rt}\Lambda \,,$$

where r is the annual interest rate and t is in years.

a. Show that the expected value of $S(t)$ is $e^{rt}S(0)$.

$$\begin{aligned} E(S(0)e^{rt}\Lambda) &= S(0)e^{rt}E[\Lambda] \\ &= S(0)e^{rt}\sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} \lambda e^{-\sigma(\lambda-1)^2}\, d\lambda \\ &= S(0)e^{rt}. \end{aligned}$$

Use the formulas above with τ equal to 1, to evaluate the integral.

b. Explain why the result of part a. is a good thing.

This result can be used to show, via a duplicating portfolio argument, that

$$e^{rt}D(0) = E[D(t)],$$

where $D(t)$ denotes a random variable, whose values are a function of the values of $S(t)$. Thus giving a formula relating the current value of the option to its future expected value, with this later being something computable.

c. Suppose you had an option D with a strike price of X whose payoff at time T equals

$$D(T) = \begin{cases} 0, & X \ge S(T) \\ 1, & X < S(T) \end{cases} .$$

Explain how you would calculate $D(0)$.

First we note that

$$\begin{aligned} e^{rT}D(0) &= E[D(T)] \\ &= \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} D(T)e^{-\sigma(\lambda-1)^2}\, d\lambda \,. \end{aligned}$$

We next need to determine when $X < S(T)$. This leads to the inequalities

$$X < S(0)e^{rT}\lambda \text{ or } e^{-rT}\frac{X}{S(0)} < \lambda \,.$$

Set $\hat{d} = e^{-rT}\frac{X}{S(0)}$ Thus, we have

$$e^{rT}D(0) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} D(T)e^{-\sigma(\lambda-1)^2}\, d\lambda$$

$$= \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\hat{d}} D(T)e^{-\sigma(\lambda-1)^2}\, d\lambda + \sqrt{\frac{\sigma}{\pi}} \int_{\hat{d}}^{\infty} D(T)e^{-\sigma(\lambda-1)^2}\, d\lambda$$

$$= \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\hat{d}} 0\; e^{-\sigma(\lambda-1)^2}\, d\lambda + \sqrt{\frac{\sigma}{\pi}} \int_{\hat{d}}^{\infty} 1\; e^{-\sigma(\lambda-1)^2}\, d\lambda$$

$$= \sqrt{\frac{\sigma}{\pi}} \int_{\hat{d}}^{\infty} e^{-\sigma(\lambda-1)^2}\, d\lambda\,.$$

d. Suppose you had a listing of stock prices $\{S_i\}_{i=0}^{N}$ for some company, and that the time interval between each stock price was the same, say Δt. How would you use this data to determine the value of σ in the random variable Λ used in modeling stock prices?

Using this stock price model the random variable S_i is obtained from the random variable S_{i-1} by

$$S_i = S_{i-1}e^{r(\Delta t)}\Lambda_i,$$

where Λ_i is a particular implementation of Λ. Thus, we have

$$\frac{S_i}{S_{i-1}} = e^{r(\Delta t)}\Lambda_i.$$

This leads to the following formula

$$\mathrm{Var}\left[\frac{S_i}{S_{i-1}}\right] = \mathrm{Var}[e^{r(\Delta t)}\Lambda_i] = e^{2r(\Delta t)}\mathrm{Var}(\Lambda_i) = e^{2r(\Delta t)}\frac{1}{2\sigma}.$$

So all we need do is to determine the expected value and variation of the data S_i/S_{i-1}, and then solve the second equation for σ. How to calculate the expected value and variation of S_i/S_{i-1} is shown below.

$$\mathrm{E}\left[\frac{S_i}{S_{i-1}}\right] = \frac{1}{N}\sum_{i=1}^{N}\frac{S_i}{S_{i-1}} = M$$

$$\mathrm{Var}\left[\frac{S_i}{S_{i-1}}\right] = \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{S_i}{S_{i-1}} - M\right)^2$$

Note: the value of M should equal $e^{r(\Delta t)}$.