Converting the Black-Scholes PDE to The Heat Equation

The Black-Scholes partial differential equation and boundary value problem is

\[
L(V) = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad 0 \leq S, \quad 0 \leq t \leq T
\]

\[
V(S,T) = f(S), \quad 0 \leq S, \quad V(0,t) = 0, \quad 0 \leq t \leq T.
\]

If \( V \) is the price of a call option, then the boundary condition \( f(S) = \max(S - E, 0) \), where \( E \) denotes the strike price of the call option.

The following change of variables transforms the Black-Scholes boundary value problem into a standard boundary value problem for the heat equation.

\[
S = e^x, \quad t = T - \frac{2\tau}{\sigma^2},
\]

\[
V(S,t) = v(x,\tau) = v \left( \ln(S), \frac{\sigma^2}{2}(T - t) \right).
\]

The partial derivatives of \( V \) with respect to \( S \) and \( t \) expressed in terms of partial derivatives of \( v \) in terms of \( x \) and \( \tau \) are:

\[
\frac{\partial V}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial v}{\partial \tau}
\]

\[
\frac{\partial V}{\partial S} = \frac{1}{S} \frac{\partial v}{\partial x}
\]

\[
\frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2} \frac{\partial v}{\partial x} + \frac{1}{S^2} \frac{\partial^2 v}{\partial x^2}
\]

Placing these expressions into the Black-Scholes partial differential equation and simplifying we have

\[
\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + \left( \frac{2r}{\sigma^2} - 1 \right) \frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v.
\]

Setting \( \kappa = \frac{2r}{\sigma^2} \) and \( t = \tau \), the Black-Scholes boundary value problem becomes

\[
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (\kappa - 1) \frac{\partial v}{\partial x} - \kappa v, \quad -\infty < x < \infty, \quad 0 \leq t \leq \frac{\sigma^2}{2} T
\]

\[
v(x,0) = V(e^x,T) = f(e^x), \quad -\infty < x < \infty
\]
One more change of variables is needed in order to eliminate the last two terms on the right hand side of the last equation. To this end set

\[ v(x, t) = e^{\alpha x + \beta t} u(x, t) = \phi u, \]

where we’ll pick \( \alpha \) and \( \beta \) later. Computing the partials of \( v \) in terms of \( x \) and \( t \) we have

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \beta \phi u + \phi \frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial x} &= \alpha \phi u + \phi \frac{\partial u}{\partial x} \\
\frac{\partial^2 v}{\partial x^2} &= \alpha^2 \phi u + 2 \alpha \phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}
\end{align*}
\]

Placing these expressions into the partial differential equation which \( v \) satisfies, and setting

\[
\alpha = -\frac{1}{2} (k - 1) = \frac{\sigma^2 - 2r}{2\sigma^2}
\]

\[
\beta = -\frac{1}{4} (k + 1)^2 = -\left( \frac{\sigma^2 + 2r}{2\sigma^2} \right)^2.
\]

we have

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < t \leq \frac{\sigma^2}{2} T \\
u(x, 0) &= e^{-\alpha x} v(x, 0) = e^{-\alpha x} f(e^x), \quad -\infty < x < \infty
\end{align*}
\]

(1)

(2)

If the option is a call option, with strike price \( E \), then \( f(x) = \max(x - E, 0) \), and

\[
u(x, 0) = e^{-\alpha x} \max(e^x - E, 0).
\]

It can be shown that the solution to the heat equation (1) and initial condition (2) is given by the following integral

\[
u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(\xi, 0) e^{-\frac{(x-\xi)^2}{4t}} \, d\xi.
\]
Find the value of an option, whose value at expiration equals \( f(S) \), where

\[
f(S) = \begin{cases} 
0, & S < 1 \\
3, & 1 \leq S \leq 2 \\
0, & S > 3 
\end{cases}
\]

\[
V(S, 0) = \nu \left( \ln S, \frac{\sigma^2 T}{2} \right) = e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} u \left( \ln S, \frac{\sigma^2 T}{2} \right)
\]

\[
= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{1}{\sqrt{4\pi \frac{\sigma^2 T}{2}}} \int_{-\infty}^{\infty} u(\xi, 0) e^{-\frac{(\ln S - \xi)^2}{2 \frac{\sigma^2 T}{2}}} d\xi
\]

\[
= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{1}{\sqrt{2\pi \sigma^2 T}} \int_{-\infty}^{\infty} e^{-\alpha \xi} f(e^\xi) e^{-\frac{(\ln S - \xi)^2}{2 \frac{\sigma^2 T}{2}}} d\xi
\]

\[
= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{3}{\sqrt{2\pi \sigma^2 T}} \int_{0}^{\ln 2} e^{-\alpha \xi} e^{-\frac{(\ln S - \xi)^2}{2 \frac{\sigma^2 T}{2}}} d\xi
\]

\[
= e^{\alpha \ln S} e^{\beta \frac{\sigma^2 T}{2}} \frac{3S^{-\alpha}}{\sqrt{2\pi \sigma^2 T}} e^{\frac{\sigma^2 T}{2}} \int_{\lambda_1}^{\lambda_2} e^{-\lambda^2/2} d\lambda
\]

\[
\lambda_1 = \frac{\ln(S/2) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \\
\lambda_2 = \frac{\ln(S + (r - \sigma^2/2)T)}{\sigma \sqrt{T}}
\]

\[
= 3e^{-\frac{\sigma^2 + 8r T}{8}} \left[ N(\lambda_2) - N(\lambda_1) \right].
\]