Brownian Motion and Stochastic Differential Equations
Math 425

1 Brownian Motion

Mathematically Brownian motion, \( B_t \) \( 0 \leq t \leq T \), is a set of random variables, one for each value of the real variable \( t \) in the interval \([0, T]\). This collection has the following properties:

- \( B_t \) is continuous in the parameter \( t \), with \( B_0 = 0 \).
- For each \( t \), \( B_t \) is normally distributed with expected value 0 and variance \( t \), and they are independent of each other.
- For each \( t \) and \( s \) the random variables \( B_{t+s} - B_s \) and \( B_s \) are independent. Moreover \( B_{t+s} - B_s \) has variance \( t \).

The above states some properties, which Brownian motion must possess. However, just because we want something with certain properties does not guarantee that such a thing exists.

It can be shown that Brownian motion does indeed exist, and section 5.9 of The Mathematics of Finance Modeling and Hedging by Stampfli and Goodman indicates one way to construct a Brownian motion.

There is one important fact about Brownian motion, which is needed in order to understand why the process

\[ S_t = e^{\sigma B_t e^{(\mu - \sigma^2/2)t}} \]

satisfies the stochastic differential equation

\[ dS = \mu S dt + \sigma S dB. \]  

The crucial fact about Brownian motion, which we need is

\[ (dB)^2 = dt. \]  

Equation (3) says two things. First \((dB)^2\) is determinant, it is not random, and it’s magnitude is \(dt\). That is, the amount of change in \((dB)^2\) caused by a change \(dt\) in the parameter is equal to \(dt\).

To partially justify this statement we compute the expected value of \((B_{t+\Delta} - B_t)^2\).

\[ E[(B_{t+\Delta} - B_t)^2] = \text{Var}[(B_{t+\Delta} - B_t)] \]
\[ = \Delta. \]

2 Ito’s Lemma

For a function \(f(x, y)\) of the variables \(x\) and \(y\) it is not at all hard to justify that the equation below is correct to first order terms.

\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \]
However, what if we have a function $f$ which depends not only on a real variable $t$, but also on a stochastic process such as Brownian motion. Suppose that $f = f(t, B_t)$, where $B_t$ denotes Brownian motion. One is tempted to write as before that

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t. \quad (7)$$

However, in this case we would be badly mistaken. To see that this is so, we expand $df$ using Taylor’s formula; this time keep the terms involving the second derivatives of $f$

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 + \frac{\partial^2 f}{\partial t \partial B_t} dt dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial B_t^2} (dB_t)^2 + \text{higher order terms}. \quad (8)$$

We discard all terms involving $dt$ to a power higher than 1. Note that the term $dt dB_t$ has magnitude $(dt)^{3/2}$. This leaves the following expression for $df$:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial B_t^2} (dB_t)^2. \quad (9)$$

We next use the fact that $(dB_t)^2 = dt$ and write

$$df = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B_t^2} \right) dt + \frac{\partial f}{\partial B_t} dB_t. \quad (10)$$

Equation (10) is called Ito’s lemma, and gives us the correct expression for calculating differentials of composite functions which depend on Brownian processes.

### 3 Applications of Ito’s Lemma

Let $f(B_t) = B_t^2$. Then Ito’s lemma gives

$$d \left( B_t^2 \right) = dt + 2B_t dB_t$$

This formula leads to the following integration formula

$$\int_{t_0}^t B_\tau dB_\tau = \frac{1}{2} \left( \int_{t_0}^t d \left( B_\tau^2 \right) - \int_{t_0}^t d\tau \right) = \frac{B_t^2 - B_{t_0}^2}{2} - t - t_0. \quad (11)$$

Contrast this formula with the normal version

$$\int x \, dx = \frac{x^2}{2}.$$

Verify that $S_t = e^{\sigma B_t} e^{(\mu - \sigma^2/2)t}$ satisfies the stochastic differential equation

$$dS = \mu S dt + \sigma S dB_t. \quad (11)$$