Spring, 2012

Brownian Motion and Stochastic Differential Equations

Math 425

1 Brownian Motion

Mathematically Brownian motion, B_t $0 \le t \le T$, is a set of random variables, one for each value of the real variable t in the interval $[0, T]$. This collection has the following properties:

- B_t is continuous in the parameter t, with $B_0 = 0$.
- For each t, B_t is normally distributed with expected value 0 and variance t, and they are independent of each other.
- For each t and s the random variables $B_{t+s} - B_s$ and B_s are independent. Moreover $B_{t+s} - B_s$ has variance t.

The above states some properties, which Brownian motion must possess. However, just because we want something with certain properties does not guarantee that such a thing exists.

It can be shown that Brownian motion does indeed exist, and section 5.9 of The Mathematics of Finance *Modeling and Hedging* by Stampfli and Goodman indicates one way to construct a Brownian motion.

There is one important fact about Brownian motion, which is needed in order to understand why the process

$$S_t = e^{\sigma B_t} e^{(\mu - \sigma^2/2)t} \tag{1}$$

satisfies the stochastic differential equation

$$dS = \mu S dt + \sigma S dB \,. \tag{2}$$

The crucial fact about Brownian motion, which we need is

$$(dB)^2 = dt \,. \tag{3}$$

Equation (3) says two things. First $(dB)^2$ is determinant, it is not random, and it's magnitude is dt. That is, the amount of change in $(dB)^2$ caused by a change dt in the parameter is equal to dt.

To partially justify this statement we compute the expected value of $(B_{t+\triangle} - B_t)^2$.

$$\begin{aligned} \mathrm{E}\left[(B_{t+\triangle} - B_t)^2\right] &= \mathrm{Var}\left[(B_{t+\triangle} - B_t)\right] \quad (4) \\ &= \triangle \,. \quad (5) \end{aligned}$$

2 Ito's Lemma

For a function $f(x, y)$ of the variables x and y it is not at all hard to justify that the equation below is correct to first order terms.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \,. \tag{6}$$

However, what if we have a function f which depends not only on a real variable t, but also on a stochastic process such as Brownian motion. Suppose that $f = f(t, B_t)$, where B_t denotes Brownian motion. One is tempted to write as before that

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t \,. \tag{7}$$

However, in this case we would be badly mistaken. To see that this is so, we expand df using Taylor's formula; this time keep the terms involving the second derivatives of f

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial B_t} dt dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2}(dB_t)^2 + \text{ higher order terms.} \tag{8}$$

We discard all terms involving dt to a power higher than 1. Note that the term $dtdB_t$ has magnitude $(dt)^{3/2}$. This leaves the following expression for df:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B_t} dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2}(dB_t)^2 \,. \tag{9}$$

We next use the fact that $(dB_t)^2 = dt$ and write

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2} \right) dt + \frac{\partial f}{\partial B_t} dB_t \,. \tag{10}$$

Equation (10) is called Ito's lemma, and gives us the correct expression for calculating differentials of composite functions which depend on Brownian processes.

3 Applications of Ito's Lemma

Let $f(B_t) = B_t^2$. Then Ito's lemma gives

$$d\left(B_t^2\right) = dt + 2B_t \, dB_t$$

This formula leads to the following integration formula

$$\begin{aligned} \int_{t_0}^{t} B_\tau \, dB_\tau &= \frac{1}{2}\left(\int_{t_0}^{t} d\left(B_\tau^2\right) - \int_{t_0}^{t} d\tau \right) \\ &= \frac{B_t^2 - B_{t_0}^2}{2} - \frac{t - t_0}{2} \,. \end{aligned}$$

Contrast this formula with the normal version

$$\int x \, dx = \frac{x^2}{2} \,.$$

Verify that $S_t = e^{\sigma B_t} e^{(\mu - \sigma^2/2)t}$ satisfies the stochastic differential equation

$$dS = \mu S dt + \sigma S dB_t \,. \tag{11}$$