Forward Contracts

There are two issues regarding forward contracts:

1. What is a 'fair' strike price?
2. At a time after the contract is written, but before the expiration date, what is the contract worth?

We assume in the following that the current interest rate is $100r\%$ per year, and that it is compounded continuously. To determine a fair strike price we use the principle of no arbitrage. Let S_0 denote the value of the asset when the contract is initiated. Let X denote the strike price and let T denote the time in years to the strike (expiration) date. Then a fair (*arbitrage free*) strike price is

$$X = S_0 e^{rT} .$$

To establish this formula we will show that if equality does not hold then an arbitrage opportunity exists.

Case: $X < S_0 e^{rT}$

Short sell the asset and 'buy' the forward contract, then invest the money obtained by shorting the asset. At time T you will have $S_0 e^{rT}$ dollars. Take X of them to buy the asset, thus satisfying the forward contract, and then use the asset to close out your short position. You will be left with a risk free profit of $S_0 e^{rT} - X > 0$ dollars. Thus, we must have $X \geq S_0 e^{rT}$.

Case: $X > S_0 e^{rT}$

If this situation arises, then short the forward contract, borrow S_0 dollars and use it to buy the asset. Then at time T do the following: close out the forward contract by turning over the asset and receive the strike price of X. Use this money to pay back the loan and you are left with a risk free profit of $X - S_0 e^{rT} > 0$ dollars. Thus, the arbitrage free strike price must be $S_0 e^{rT}$.

We next want to determine the value of a forward contract. Let T denote the time, in years, from inception of the contract to its expiration. Let $f_t,\ 0 \leq t \leq T$ denote the value of the contract, to the holder, at time t years after the contract was initiated. Let S_t denote the value of the asset at time t, and, as usual, let the continuously compounded yearly interest rate be $100r\%$, and finally let X denote the contract's strike price.

For the holder of the forward contract its value f_t satisfies:

$$f_0 = 0, \quad f_T = S_T - X$$

The second formula above is just the statement that the holder of this contract will pay X dollars at expiration, and receive an asset worth S_T dollars. We claim that the value of the contract at time t, to the holder of the contract, is

$$f_t = S_t - X e^{-r(T-t)}$$

The formula is verified by an arbitrage free argument.

Case: $f_t + Xe^{-r(T-t)} < S_t$

To realize some 'free' money in this case, short the asset, buy the contract for f_t dollars and invest the remaining money $S_t - f_t$, which is greater than $Xe^{-r(T-t)}$. At time T you will have $(S_t - f_t)e^{r(T-t)}$, which is more than X. Use X dollars to close out the forward contract, and then use the asset you just bought to close out your short position. You will have a net gain of $(S_t - f_t)e^{r(T-t)} - X > 0$.

Case: $f_t + Xe^{-r(T-t)} > S_t$

In this case short the contract and borrow $Xe^{-r(T-t)}$ dollars. Use this to buy the asset for S_t dollars. At time T receive X dollars and turn over the asset, which closes out the forward contract, and then use the X to repay the loan of $Xe^{-r(T-t)}$. You are left with at least $e^{r(T-t)}\left(f_t + Xe^{-r(T-t)} - S_t\right) > 0$ dollars.

Example 1. *Southwest airlines is concerned that the cost of aviation jet fuel will be increasing. On Dec. 16, 2011 the cost of one barrel of jet fuel was \$119.90. Assume an annual interest rate of* 1.25%. *They enter into a forward contract for* 5 *million barrels of jet fuel with a strike date 6 months past Dec. 16. What is the arbitrage free strike price for this forward contract?*

On Dec. 16^{th} the value of the 5 million barrels of jet fuel equaled 5.995×10^8 dollars. Thus, the strike price was

$$5.995 \times 10^8 \times e^{\frac{0.0125}{2}} = 6.032586084 \times 10^8,$$

A bit over 603 million dollars. By the way, the Air Transport Asso. said that U.S. commercial airlines purchased over 51 million gallons of jet fuel per day in 2008. That amounts to more than $\frac{51}{42} \approx 1.215$ million barrels of fuel oil per day.

Example 2. *A forward contract on 100 bushels of hard red winter wheat was written on Dec. 1, 2011, with expiration date of Sept. 1, 2012. The price of a bushel of wheat on December 1 was \$7.225 dollars. If the strike price on the forward contract is \$732.50, what was the annual interest rate on Dec. 1?*

We know that the strike price and the value of the asset are related by the formula

$$732.50 = 722.50 \times e^{\frac{3r}{4}}\,.$$

Solving this equation for r we have

$$r = \frac{4}{3}\ln\left(\frac{732.50}{722.50}\right) \approx 0.01833\,.$$

Thus, the annual interest rate was about 1.833%.

Example 3. *Assume that on Jan. 16, 2012 the price of one barrel of jet fuel is \$120.25, and that the annual interest rate is still at 1.25%. What is the value of the forward contract mentioned in Example 1?*

Jan. 16 2012 is 1/12 of a year after the contracts initiation date. Thus, the value of the forward contract is:

$$\begin{aligned} f_{1/12} = S_{1/12} - Xe^{-0.0125(5/12)} &\approx 6.0125 \times 10^8 - 6.032586084 \times 10^8 \times e^{-0.0125(5/12)} \\ &\approx 1.125 \times 10^6 \text{ dollars.} \end{aligned}$$