Put/Call Options

A European, put or call, option is like a forward contract. There is an underlying asset usually taken to be a share of stock, a strike price X, and an expiration date. At the expiration date, the holder of a call option has the right to buy a share of the asset at the strike price, while the holder of a put option has the right to sell a share of the asset at the strike price X. However, at expiration date, the holder of the option does not have to exercise the option, in contrast to a forward contract.

An American option is like a European option except the holder of an American option may exercise the option at any time before the expiration date.

At the expiration date, the value of a call option for one share of the underlying asset either American or European equals

$$C = \max(S - X, 0)\,,$$

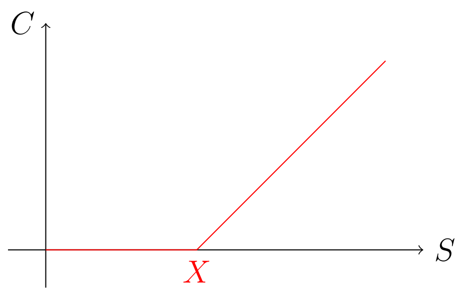
where S is the value of one share of the asset and X is the strike price of the call option.

Similarly the value of a put at expiration is

$$P = \max(X - S, 0)\,.$$

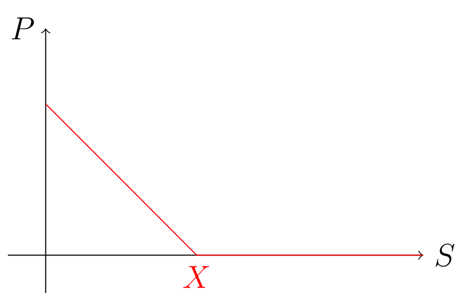
A plot of the value of a call option is shown below for various values of S.

Value of a Call



The slanted line in the plot is at 45 degrees. That is for any increase in the value of S, if $S > X$, we have the same increase in value of the call option. If $S \leq X$, then the option is worthless and would be allowed to expire with out being exercised. The plot of the value of a put option is shown below.

Value of a Put



Notice that a portfolio, which consists of a put and a call option with the same strike price and expiration date has the same value as $|S - X|$.

It turns out that the values of a put and call option for the same asset with the same strike price and expiration date are related. This relation is referred to as *Put Call Parity.*

Theorem 1. *Let C and P denote the values of a European call and put option for the same asset. Assume that both options have the same strike price X and the same expiration date. Then*

$$C - P = S - PV(X)\,,$$

where $PV(X)$ denotes the present value of the strike price X, which is to be paid at expiration.

As in other results in financial mathematics the proof utilizes the assumption that arbitrage opportunities cannot exist. We show that strict inequality in either direction results in an arbitrage opportunity, and thus we must have equaltity.

Proof:

Case: $C + PV(X) < S + P$

In this case short the asset, sell the put, buy the call and invest the amount $\mathrm{PV}(X)$. Note that there is a positive amount of money left over. At expiration we have two possibilities. Either $S > X$ or $S \leq X$.

If $S > X$, the put option will be worthless, take the X dollars earned from the investment of $\mathrm{PV}(X)$ and use it and the call option to buy the asset, then use the asset to close out your short position.

If $S \leq X$, the call option will be worthless, take the X dollars from your investment and close out the put option you sold, then take the asset bought with the put option and close out your short postition. Note that regardless of the value of S at expiration we have generated some free money.

Case: $C + PV(X) > S + P$

Finish the proof by verifying that this inequality also leads to an arbitrage opportunity.

One would think that the value of an American option would be greater than the value of a European option. While this is true for puts, it is not true for call options as the following results shows.

Theorem 2. *The value of an American call option equals the value of a European call option assuming both calls have the same strike price and expiration date.*

Proof: What we demonstrate is that it is not profitable to exercise an American call option before its expiration date. Since the only difference between an American and European option is the ability for early exercise, and it is not profitable to exercise an American call early, the two options must have the same value.

We will show that the value of an American call satisfies the following inequality:

$$C \geq S - \mathrm{PV}(X)\,,$$

where S is the value of the asset and $\text{PV}(X)$ is the present value of the strike price. Consider the following two portfolios: Portfolio I consists of buying an American call, Portfolio II consist of buying the asset and borrowing the present value of the strike price. The table below gives the values of these two portfolios at expiration.

	$S \leq X$	$S > X$
Portfolio I (buy C)	0	$S - X$
Portfolio II (buy S borrow $\text{PV}(X)$	$S - X$	$S - X$

Notice that regardless of the value of the asset at expiration Portfolio I has a value at least as great as Portfolio II. An arbitrage free argument implies that this relationship must be true at any time prior to expiration. So suppose that at some time t before expiration the holder of an American call option decides to exercise the option. Then he will realize a profit of $S - X$ dollars. But the value of the call is at least as big as $S - \text{PV}(X)$, which is greater than $S - X$. So rather than exercising the option the holder of this call will sell the call and achieve a larger return. That is, it is not profitable to early exercise an American call option.