Chapter 4

Determinants

Chapter 3 entailed a discussion of linear transformations and how to identify them with matrices. When we study a particular linear transformation we would like its matrix representation to be simple, diagonal if possible. We therefore need some way of deciding if we can simplify the matrix representation and then how to do so. This problem has a solution, and in order to implement it, we need to talk about something called the determinant of a matrix.

The determinant of a square matrix is a number. It turns out that this number is nonzero if and only if the matrix is invertible. In the first section of this chapter, different ways of computing the determinant of a matrix are presented. Few proofs are given; in fact no attempt has been made to even give a precise definition of a determinant. Those readers interested in a more rigorous discussion are encouraged to read Appendices C and D.

4.1 Properties of the Determinant

The first thing to note is that the determinant of a matrix is defined only if the matrix is square. Thus, if $A$ is a $2 \times 2$ matrix, it has a determinant, but if $A$ is a $2 \times 3$ matrix it does not. The determinant of a $2 \times 2$ matrix is now defined.

**Definition 4.1.** Determinant($A$) = det($A$) = det $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

**Example 1.** Compute the determinants of the following matrices:

a. \( \det \begin{bmatrix} 1 & 6 \\ -2 & 3 \end{bmatrix} = 3 - (-12) = 15 \)

b. \( \det \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} = 8 - 0 = 8 \)

c. \( \det \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} = 6 - 6 = 0 \)

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To compute the determinant of a $3 \times 3$ or $n \times n$ matrix, we need to introduce some notation.

**Definition 4.2.** Let $A = [a_{jk}]$ be an $n \times n$ matrix. Let $M_{jk}$ be the $(n - 1) \times (n - 1)$ matrix obtained from $A$ by deleting its $j$th row and $k$th column. This submatrix of $A$ is referred to as the $j,k$ minor of $A$.

**Example 2.** Let $A = \begin{bmatrix} -1 & 6 & 3 \\ 2 & 0 & 9 \\ 4 & 8 & 7 \end{bmatrix}$. Find $M_{11}$, $M_{23}$, and $M_{32}$.

$M_{11} = \begin{bmatrix} 0 & 9 \\ 8 & 7 \end{bmatrix}$

$M_{23} = \begin{bmatrix} -1 & 6 \\ 4 & 8 \end{bmatrix}$

$M_{32} = \begin{bmatrix} -1 & 3 \\ 2 & 9 \end{bmatrix}$

Using minors we demonstrate one way to compute the determinant of a $3 \times 3$ matrix. The technique is called expansion by cofactors.

Let $A$ be any $3 \times 3$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then

$$\det(A) = a_{11} \det(M_{11}) - a_{12} \det(M_{12}) + a_{13} \det(M_{13})$$

Note that any minor of a $3 \times 3$ matrix is a $2 \times 2$ matrix, and hence its determinant is defined. We also wish to stress that we did not have to expand across the first row. We could have used any row or column.

**Example 3.** Compute the determinant of the matrix below by expanding across the first row and also by expanding down the second column.

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 6 & 3 & 5 \\ -3 & 7 & 0 \end{bmatrix}$$

1. Expanding across the first row we have

$$\det(A) = a_{11} \det(M_{11}) - a_{12} \det(M_{12}) + a_{13} \det(M_{13})$$

$$= (-1) \det \begin{bmatrix} 6 & 5 \\ -3 & 0 \end{bmatrix} - (2) \det \begin{bmatrix} 6 & 5 \\ -3 & 0 \end{bmatrix} + (4) \det \begin{bmatrix} 6 & 3 \\ -3 & 7 \end{bmatrix}$$

$$= -(-35) - 2(15) + 4(42 + 9) = 209$$

2. Expanding down the second column we have

$$\det(A) = -a_{12} \det(M_{12}) + a_{22} \det(M_{22}) - a_{32} \det(M_{32})$$

$$= -(2) \det \begin{bmatrix} 6 & 5 \\ -3 & 0 \end{bmatrix} + (3) \det \begin{bmatrix} -1 & 4 \\ -3 & 0 \end{bmatrix} - (7) \det \begin{bmatrix} -1 & 4 \\ -3 & 5 \end{bmatrix}$$

$$= -(2)(15) + 3(12) - 7(-29) = 209$$

□
It seems from the above two computations that minus signs creep in at random. That is not true. There is a rule for deciding whether or not a minus sign should appear, and it is given in the following theorem.

**Theorem 4.1.** Let \( A = [a_{jk}] \) be any \( n \times n \) matrix. Then

\[
det(A) = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} \det(M_{jk}) \quad k = 1, 2, \ldots, n
\]

expansion down the \( k \)th column

\[
det(A) = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det(M_{jk}) \quad j = 1, 2, \ldots, n
\]

expansion across the \( j \)th row

Figure 4.1 should clarify whether or not a minus sign precedes the term \( \det(M_{jk}) \). Notice that the 1,1 entry has a + sign, and whenever we move one space horizontally or vertically the sign changes. Since it takes three moves to go from 1, 1 to 2, 3, the coefficient of \( a_{23} \) in (4.1) equals \( -\det(M_{23}) \).

The terms \( (-1)^{j+k} \det(M_{jk}) \) are called the cofactors of \( a_{jk} \), hence the phrase expansion by cofactors. Notice that this theorem reduces the problem of computing the determinant of an \( n \times n \) matrix to the problem of calculating the determinant of an \( (n-1) \times (n-1) \) matrix. Continued use of this procedure will reduce the original problem to one of calculating the determinants of \( 2 \times 2 \) matrices.

\[
\begin{bmatrix}
+ & - & + & - & + & \cdots \\
- & + & - & + & - & \cdots \\
+ & - & & & & \\
- & & & & & \\
+ & & & & & \\
- & + & - & + & \cdots \\
\vdots & & & & &
\end{bmatrix}
\]

Figure 4.1

It is clear that computing the determinant of a matrix, especially a large one, is painful. It’s also clear that the more zeros in a matrix the easier the chore. The following theorems enable us to increase the number of zeros in a matrix and at the same time keep track of how the value of the determinant changes.

**Theorem 4.2.** Let \( A \) be a square matrix. If \( A_1 \) is a matrix obtained from \( A \) by interchanging any two rows or columns, then \( \det(A_1) = -\det(A) \).
Example 4.

\[
\begin{vmatrix}
1 & 2 & 4 \\
0 & 3 & 2 \\
1 & 0 & 5 \\
\end{vmatrix}
= -
\begin{vmatrix}
0 & 3 & 2 \\
1 & 2 & 4 \\
1 & 0 & 5 \\
\end{vmatrix}
: \text{rows one and two interchanged}
\]

\[
\begin{vmatrix}
1 & 2 & 4 \\
0 & 3 & 2 \\
1 & 0 & 5 \\
\end{vmatrix}
= -
\begin{vmatrix}
1 & 4 & 2 \\
0 & 2 & 3 \\
1 & 5 & 0 \\
\end{vmatrix}
: \text{columns two and three interchanged}
\]

□

Corollary 4.1. If an \( n \times n \) matrix has two identical rows or columns, its determinant must equal zero.

Proof. The preceding theorem says that if you interchange any two rows or columns, the determinant changes sign. But if the two rows interchanged are identical, the determinant must remain unchanged. Since zero is the only number equal to its negative, we have \( \det(A) = 0 \).

□

Example 5.

a. \( \det \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0 \)

b. \( \det \begin{vmatrix} 1 & -6 & 1 \\ 2 & 3 & 8 \\ 1 & -6 & 1 \end{vmatrix} = 0: \text{rows one and three are identical} \)

□

Theorem 4.3. If any row or column of a square matrix \( A \) is multiplied by a constant \( c \) to get a matrix \( A_1 \), then \( \det(A_1) = c[\det(A)] \).

Corollary 4.2. If a square matrix \( A \) has a row or column of zeros, then \( \det(A) = 0 \).

Example 6.

\[
\begin{vmatrix}
3 & 4 & 12 \\
6 & 16 & 30 \\
9 & 8 & 21 \\
\end{vmatrix}
= \det \begin{vmatrix}
3(1) & 4 & 12 \\
3(2) & 16 & 30 \\
3(3) & 8 & 21 \\
\end{vmatrix}
= 3 \det \begin{vmatrix}
1 & 4 & 12 \\
2(1) & 2(8) & 2(15) \\
3 & 8 & 21 \\
\end{vmatrix}
= 3(1) \det \begin{vmatrix}
1 & 4 \\
2 & 8 \\
3 & 21 \\
\end{vmatrix}
= 6 \det \begin{vmatrix}
1 & 4 & 3(4) \\
1 & 8 & 3(5) \\
3 & 8 & 3(7) \\
\end{vmatrix}
= 18 \det \begin{vmatrix}
1 & 4(1) & 4 \\
1 & 4(2) & 5 \\
3 & 4(2) & 7 \\
\end{vmatrix}
= 72 \det \begin{vmatrix}
1 & 1 & 4 \\
1 & 2 & 5 \\
3 & 2 & 7 \\
\end{vmatrix}
\]

□
Theorem 4.4. If any multiple of a row (column) is added to another row (column) of a square matrix $A$ to get another matrix $A_1$, then $\det(A_1) = \det(A)$.

Example 7.

\[
\det \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 5 \\ 3 & 2 & 7 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = -4 \quad \square
\]

This last example illustrates perhaps the easiest way to evaluate the determinant of a matrix. That is, use the elementary row or column operations to get a row or column with at most one nonzero entry and then use Theorem 4.1.

Our next example also demonstrates this idea. The reader might try computing the determinant in this example by using Theorem 4.1 directly and then comparing the two techniques.

Example 8.

\[
\det \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & -8 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 9 \quad \square
\]

Theorem 4.5. Let $A$ and $B$ be two $n \times n$ matrices. Then

\[\det(AB) = \det(A) \det(B)\]

Example 9. Let $A = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 8 \end{bmatrix}$ Verify Theorem 4.5 for these two matrices.

Solution.

\[
\det(A) = \det \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix} = -10 \quad \det(B) = \det \begin{bmatrix} 1 & 3 \\ -2 & 8 \end{bmatrix} = 14
\]

\[
\det(AB) = \det \begin{bmatrix} -5 & 41 \\ 5 & -13 \end{bmatrix} = -140 = (-10)(14) = \det(A) \det(B) \quad \square
\]

Theorem 4.6. Let $A$ be any square matrix and let $A^T$ be its transpose, then

\[\det(A) = \det(A^T)\]
Example 10. Verify Theorem 4.6 for the matrix \( A = \begin{bmatrix} 6 & 4 \\ -2 & 3 \end{bmatrix} \).

**Solution.**

\[
\det(A) = \det \begin{bmatrix} 6 & 4 \\ -2 & 3 \end{bmatrix} = 18 - (-8) = 26
\]

\[
\det(A^T) = \det \begin{bmatrix} 6 & -2 \\ 4 & 3 \end{bmatrix} = 18 - (-8) = 26 = \det(A)
\]

\( \square \)

**Theorem 4.7.** A square matrix \( A \) is invertible if and only if \( \det(A) \) is nonzero.

This last theorem is one that we use repeatedly in the remainder of this text. For example, in the next section we discuss how to compute the inverse of a matrix in terms of the determinants of its minors, and in Chapter 5 we use an equivalent version of Theorem 4.7 that says, if \( \ker(A) \) has nonzero vectors in it, then \( \det(A) = 0 \).

**Problem Set 4.1**

1. Compute the determinants of each of the following matrices:
   a. \( \begin{bmatrix} -6 & 0 \\ 1 & 2 \end{bmatrix} \)  
   b. \( \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \)  
   c. \( \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ -1 & 4 & 1 \end{bmatrix} \)  
   d. \( \begin{bmatrix} -1 & 2 & 6 & 4 \\ 1 & 0 & 2 & 8 \\ 0 & 3 & 9 & 6 \\ 2 & 7 & 5 & 6 \end{bmatrix} \)

2. Compute the determinants of each of the following matrices:
   a. \( \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \)  
   b. \( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \)  
   c. \( \begin{bmatrix} 1 & 0 & -2 \\ 4 & 6 & 0 \\ 1 & 1 & 0 \end{bmatrix} \)  
   d. \( \begin{bmatrix} 1 & 4 & 1 \\ -2 & 0 & 0 \end{bmatrix} \)

3. a. \( \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \)  
   b. \( \det \begin{bmatrix} a_1 & a_2 & a_3 \\ a_1 + k & a_2 + k & a_3 + k \end{bmatrix} \)  
   \( \begin{bmatrix} a_1 + 2k & a_2 + 2k & a_3 + 2k \end{bmatrix} \)

4. a. \( \det \begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 1 \end{bmatrix} \)  
   b. \( \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix} \)  
   \( \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} \)

5. If \( E \) is an elementary row matrix associated with adding a multiple of one row to another, then \( \det(E) = ? \)

6. Let \( A \) be any upper or lower triangular matrix. Show that \( \det(A) = a_{11}a_{22} \ldots a_{nn} \); that is, the determinant of \( A \) equals the product of the diagonal entries of \( A \).
4.1. PROPERTIES OF THE DETERMINANT

7. Let $a_j, j = 1, \ldots, n$ be arbitrary numbers and let $k$ be any number. We assume below that $n > 2$.

a. Show that

$$
\begin{vmatrix}
a_1 & a_2 & \cdots & a_n \\
a_1 + k & a_2 + k & \cdots & a_n + k \\
a_1 + 2k & a_2 + 2k & \cdots & a_n + 2k \\
\vdots & \vdots & \ddots & \vdots \\
a_1 + (n-2)k & a_2 + (n-2)k & \cdots & a_n + (n-2)k \\
a_1 + (n-1)k & a_2 + (n-1)k & \cdots & a_n + (n-1)k \\
\end{vmatrix} = 0
$$

b. Show that

$$
\begin{vmatrix}
1 & 2 & \cdots & n \\
n + 1 & n + 2 & \cdots & 2n \\
2n + 1 & 2n + 2 & \cdots & 2n + n \\
\vdots & \vdots & \ddots & \vdots \\
n(n-1)+1 & \cdots & n^2 \\
\end{vmatrix} = 0
$$

c. What happens if $n = 2$?

8. Let $A_2 = \begin{bmatrix} 0 & a_2 \\ a_1 & 0 \end{bmatrix}$. Let $A_3 = \begin{bmatrix} 0 & 0 & a_3 \\ 0 & a_2 & 0 \\ a_1 & 0 & 0 \end{bmatrix}$, and

$$
A_n = \begin{bmatrix}
0 & 0 & \cdots & a_n \\
0 & 0 & \cdots & a_{n-1} & 0 \\
0 & a_2 & 0 & \cdots & 0 \\
a_1 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}.
$$
Thus, $A_n$ is an $n \times n$ matrix whose only possible nonzero entries occur in the $(n-i+1)$st column of the $i$th row. Show that $\det(A_n) = (-1)^{n(n-1)/2}a_1a_2\ldots a_n$.

9. Let $x_1, x_2, \ldots, x_n$ be any $n$ numbers. Let

$$
\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} = A_2
$$

and

$$
\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} = A_2
$$

$$
\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix} = A_n
$$

The reader should note that these matrices (actually their transposes) appeared in Section 3.4, when we discussed fitting polynomials to prescribed data. The determinants in this problem are called Vandermonde determinants.
a. Show that \( \det(A_2) = x_2 - x_1 \).

b. \( \det(A_3) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) \).

c. Find a similar formula for \( \det(A_n) \).

10. Let \( A \) be any \( n \times n \) matrix. Show that \( \det(A) \) equals zero if and only if the rows (columns) of \( A \) form a linearly dependent set of vectors in \( \mathbb{R}^n \).

(Hint: Use Theorems 4.4 and 4.7.)

11. Let \( A \) be a nonsingular matrix. Show that \( \det(A^{-1}) = [\det(A)]^{-1} \). (Hint: \( AA^{-1} = I_n \). Now use Theorem 4.5.)

12. Let \( S \) be any scalar matrix, that is, \( S = cI_n \) for some number \( c \). Show that \( \det(S) = c^n \).

13. Let \( A \) and \( B \) be any two similar matrices; that is, there is a nonsingular matrix \( P \) such that \( A = PBP^{-1} \). Show that \( \det(A) = \det(B) \).

14. Compute the determinants of the following matrices:

\[
\begin{pmatrix}
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 \\
5 & 6 & 7 & 8
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-8 & 7 & 6 & -1 \\
2 & 5 & 4 & 3 \\
1 & -5 & 7 & 11 \\
0 & 2 & -3 & 6
\end{pmatrix}
\]

15. A matrix is said to be skew symmetric if \( A = -A^T \). Thus \( \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \) is skew symmetric, while \( \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \) is not.

a. Show that all the diagonal elements of a skew symmetric matrix are equal to zero.

b. Let \( A \) be a \( 3 \times 3 \) skew symmetric matrix. Show that \( \det(A) = 0 \).

c. If \( A \) is a \( 2 \times 2 \) skew symmetric matrix, must \( \det(A) = 0 \)?

d. What happens if \( A \) is an \( n \times n \) skew symmetric matrix?

16. Let \( P \) be an \( n \times n \) permutation matrix. Show that \( \det(P) = \pm 1 \). For the definition of a permutation matrix see problem 13 in Problem Set 1.5.

4.2 The Adjoint Matrix and \( A^{-1} \)

Theorem 4.7 states that \( A \) is invertible if and only if \( \det(A) \) is nonzero. In this section we show how to compute the inverse of a matrix by using the determinants of its \( (n - 1) \times (n - 1) \) minors (cf. Definition 4.2). Since these determinants will appear quite frequently, we introduce a special notation for them.
4.2. THE ADJOINT MATRIX AND $A^{-1}$

**Definition 4.3.** Let $A = [a_{jk}]$ be any $n \times n$ matrix. The cofactor of $a_{jk}$, denoted by $A_{jk}$, is defined to be

$$A_{jk} = (-1)^{j+k} \det(M_{jk})$$

where $M_{jk}$ is the $j,k$ minor of $A$.

**Example 1.** Let $A = \begin{bmatrix} -1 & 6 & 3 \\ 2 & 0 & 9 \\ 4 & 8 & 7 \end{bmatrix}$.

- $A_{11} = (-1)^2(-72) = (14 - 36) = (16) = (12)$
- $A_{12} = (-1)^3(14 - 36) = (14 - 36) = (15) = (12)$
- $A_{13} = (-1)^4(16) = (16)$
- $A_{21} = (-1)(42 - 24) = (18)$
- $A_{22} = (-7 - 12) = (-19) = (15)$
- $A_{23} = (-8 - 24) = (-32) = (16)$
- $A_{31} = (54)$
- $A_{32} = (-9 - 6) = (-15) = (15)$
- $A_{33} = (-12)$

Using the cofactors of a matrix, we construct its adjoint matrix.

**Definition 4.4.** Let $A = [a_{jk}]$ be an $n \times n$ matrix. The adjoint matrix of $A$ is the following $n \times n$ matrix:

$$\text{adj}(A) = [A_{jk}]^T$$

where $A_{jk}$ are the cofactors of $A$.

Note that the transpose of the matrix $[A_{jk}]$ must be taken.

**Example 2.** Let $A = \begin{bmatrix} -1 & 6 & 3 \\ 2 & 0 & 9 \\ 4 & 8 & 7 \end{bmatrix}$. From Example 1 and the definition of $\text{adj}(A)$ we have

$$\text{adj}(A) = \begin{bmatrix} -72 & 22 & 16 \\ -18 & -19 & 32 \\ 54 & 15 & -12 \end{bmatrix}^T = \begin{bmatrix} -72 & -18 & 54 \\ 22 & -19 & 15 \\ 16 & 32 & -12 \end{bmatrix} \square$$

**Example 3.** Let $A$ be the matrix in the preceding example. Compute $\det(A)$ and $A[\text{adj}(A)]$.

$$\det(A) = \det \begin{bmatrix} -1 & 6 & 3 \\ 2 & 0 & 9 \\ 4 & 8 & 7 \end{bmatrix} = \det \begin{bmatrix} 0 & 12 & 15 \\ 0 & 32 & 19 \end{bmatrix} = 252$$

$$A[\text{adj}(A)] = \begin{bmatrix} 0 & 12 & 15 \\ 0 & 32 & 19 \end{bmatrix} = \begin{bmatrix} 252 & 0 & 0 \\ 0 & 252 & 0 \end{bmatrix}$$

$$= 252I_3 = \det(A)I_3 \square$$
**Example 4.** Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Compute \( \det(A) \) and \( A \text{adj}(A) \).

**Solution.** We exhibit the minors of \( A \) first, and then compute the adjoint of \( A \).

\[
M_{11} = d \quad M_{12} = c \\
M_{21} = d \quad M_{22} = a
\]

Thus,

\[
\text{adj}(A) = \text{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

\[
A \text{adj}(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}
\]

\[
= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \det(A)I_2. \quad \Box
\]

The preceding two examples showed that \( A \text{adj}(A) = \det(A)I \) for any \( 2 \times 2 \) matrix and one particular \( 3 \times 3 \) matrix. The next theorem states that this formula is true for any square matrix.

**Theorem 4.8.** Let \( A \) be any \( n \times n \) matrix. Then

\[
A[\text{adj}(A)] = [\text{adj}(A)]A = [\det(A)]I_n
\]

**Corollary 4.3.** Let \( A \) be any square matrix with nonzero determinant. Then \( A \) is nonsingular and

\[
A^{-1} = (\det(A))^{-1}\text{adj}(A)
\]

**Example 5.** Let \( A = \begin{bmatrix} -1 & 6 & 3 \\ 2 & 0 & 9 \\ 4 & 8 & 7 \end{bmatrix} \). In Examples 2 and 3, we computed the determinant and adjoint of this matrix. Using those results, we have

\[
A^{-1} = (\det(A))^{-1}\text{adj}(A) = (252)^{-1} \begin{bmatrix} -72 & -18 & 54 \\ 22 & -19 & 15 \\ 16 & 32 & -12 \end{bmatrix} \quad \Box
\]

**Example 6.** Let \( A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ -1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 2 & -1 & 0 & 0 \end{bmatrix} \). Compute the adjoint of \( A \) and verify Theorem 4.8.

\[
\text{adj}(A) = \begin{bmatrix} 5 & 10 & -3 & 1 \\ -1 & -2 & -12 & 4 \\ -1 & -2 & 9 & 4 \\ 5 & -11 & -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -1 & 5 \\ 10 & -2 & -2 & -11 \\ -3 & -12 & 9 & -3 \\ 1 & 4 & 4 & 1 \end{bmatrix}
\]
4.2. THE ADJOINT MATRIX AND $A^{-1}$

A quick computation shows that

$$A \text{adj}(A) = \begin{bmatrix} 21 & 0 & 0 & 0 \\ 0 & 21 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix} = \det(A)I_4 \quad \square$$

Problem Set 4.2

1. Compute the adjoint matrix of each of the following matrices and verify that $A \text{adj}(A) = \det(A)I_n$:
   a. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
   b. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
   c. $\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}$
   d. $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$

2. Compute the adjoint matrix of each of the following matrices and verify that $A \text{adj}(A) = \det(A)I_n$:
   a. $\begin{bmatrix} -2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 0 & 8 \\ -2 & 3 & 0 \end{bmatrix}$
   c. $\begin{bmatrix} 2 & -1 & -2 \\ 1 & 0 & -4 \\ 0 & -1 & 6 \end{bmatrix}$

3. Using Corollary 4.3, compute the inverse of each of the nonsingular matrices in problem 1.

4. Using Corollary 4.3, compute the inverse of each of the nonsingular matrices in problem 2.

5. Consider the following system of equations:

   \begin{align*}
   x_1 - x_2 + x_3 - x_4 &= 1 \\
   x_1 + x_2 - x_3 + x_4 &= 2 \\
   x_1 + x_2 + x_3 - x_4 &= 3 \\
   x_1 + x_2 + x_3 + x_4 &= 4
   \end{align*}

   a. Solve this system using Gaussian elimination.
   b. Using Corollary 4.3, compute the inverse of the coefficient matrix $A$.
   c. Using $A^{-1}$, solve this system.

6. Suppose $A$ is a $2 \times 2$ upper triangular matrix. Show that $\text{adj}(A)$ is also upper triangular. Is this also true for $3 \times 3$ matrices?

7. Let $A$ be a square matrix.
   a. If $A$ is invertible, show that $\text{adj}(A)$ is also invertible. Find a formula for $\det(\text{adj}(A))$ and for the inverse of $\text{adj}(A)$.
   b. If $A$ is not invertible, show that $\text{adj}(A)$ is not invertible. (Hint: Theorem 4.8.) Thus, $\det(A) = 0$ if and only if $\det(\text{adj}(A)) = 0$.

8. How are the adjoints of $A$ and $A^T$ related?
4.3 Cramer’s Rule

Cramer’s rule is a formula for computing the solution to a system of linear equations when the coefficient matrix $A$ is nonsingular. This formula is just the component version of the equation

$$A^{-1} = (\det(A))^{-1} \text{adj}(A) \quad (4.2)$$

We derive Cramer’s rule for a system of two equations with two unknowns before stating the rule for $n$ equations with $n$ unknowns.

**Example 1.** Find the solution to

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad (4.3)$$

Assuming that $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$,

$$A^{-1} = (\det(A))^{-1} \text{adj}(A) = (\det A)^{-1} \begin{bmatrix} a_{22} & -a_{22} \\ -a_{21} & a_{11} \end{bmatrix}$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (\det A)^{-1} \begin{bmatrix} a_{22} & -a_{22} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1a_{22} - b_2a_{12} \\ b_2a_{11} - b_1a_{21} \end{bmatrix} (\det A)^{-1}$$

Thus,

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{\det(A)} = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det(A)}$$

$$x_2 = \frac{b_2a_{11} - b_1a_{21}}{\det(A)} = \frac{\det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det(A)} \quad \square$$

Thus we see that $x_1$ can be expressed as the ratio of two determinants. The matrix in the numerator is obtained by replacing the first column of $A$ with the right-hand side of (4.3). To calculate $x_2$, the matrix in the numerator is obtained by replacing the second column of $A$ with the right-hand side of (4.3).

Consider the general system with $n$ equations and $n$ unknowns

$$Ax = b \quad (4.4)$$

where $A = [a_{jk}]$ is an $n \times n$ nonsingular matrix $x = [x_1, \ldots, x_n]^T$, and $b = [b_1, \ldots, b_n]^T$. We know that

$$x = A^{-1}b = (\det A)^{-1} \text{adj}(A)b \quad (4.5)$$
Using (4.5), we write $x_k$ in terms of $b$:

$$x_k = (\det A)^{-1}(A_{1k}b_1 + A_{2k}b_2 + \cdots + A_{nk}b_n)$$  \hspace{1cm} (4.6)

remember $\text{adj}(A) = [A_{jk}]^T$, where $A_{jk}$ is the cofactor of $a_{jk}$. The last factor in (4.6) can be thought of as the expansion by minors (going down the $k$th column) of the determinant of the matrix obtained by replacing the $k$th column of $A$ with the column $[b_1, \ldots, b_n]^T$. That is,

$$x_k = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1(k-1)} & b_1 & a_{1(k+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{j1} & \cdots & b_j & \cdots & a_{jn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(k-1)} & b_n & a_{n(k+1)} & \cdots & a_{nn} \end{vmatrix}}{\det(A)}$$  \hspace{1cm} (4.7)

Formula (4.7) is Cramer’s rule. Note that it is valid only for systems whose coefficient matrix is nonsingular.

**Example 2.** Find the value of $x_4$ for which $x_1, x_2, x_3, x_4$, and $x_5$ solve the following system:

$$
\begin{align*}
2x_1 + x_2 &+ 2x_5 = 2 \\
x_1 + x_4 &= 2 \\
3x_2 + x_3 + 2x_4 &= -1 \\
x_1 + x_2 + x_4 + x_5 &= -1 \\
x_3 + x_4 + x_5 &= -1
\end{align*}
$$

An easy computation shows that the coefficient matrix $A$ has a determinant equal to $-7$. Thus, $A$ is nonsingular and we have

$$x_4 = \frac{\begin{vmatrix} 2 & 1 & 0 & 2 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 3 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{vmatrix}}{\det(A)} = -\left(\frac{13}{7}\right)$$

We now have three techniques for solving systems of equations: Gaussian elimination, which always works; inversion of the coefficient matrix $A$, i.e., compute $A^{-1}$; and Cramer’s rule. The last two methods assume, of course, that $A$ is invertible. For small systems, $n = 2, 3, \text{ or } 4$, any of these methods would be fine, at least if $A$ is invertible. For large $n$, however, the most efficient method to use in solving a system of equations is usually Gaussian elimination. It is for this reason that Cramer’s rule is a theoretical rather than a problem-solving tool.
Problem Set 4.3

1. Use Cramer’s rule to solve each of the following systems of equations:
   a. $2x_1 + 2x_2 = 7$
      $8x_1 + x_2 = -2$
   b. $-8x_1 + 6x_2 = 4$
      $3x_1 + 2x_2 = 6$

2. Use Gaussian elimination to solve each of the systems in problem 1.

3. Use Cramer’s rule to solve the following system:
   
   \[
   \begin{align*}
   2x_1 - 6x_2 + x_3 &= 2 \\
   x_2 + x_3 &= 1 \\
   x_1 - x_2 - x_3 &= 0
   \end{align*}
   \]

4. Consider the following system of equations:
   
   \[
   \begin{align*}
   2x_1 + x_2 + x_3 - x_4 &= 1 \\
   3x_1 - x_2 + x_3 + x_4 &= 0 \\
   -x_1 + x_2 - x_3 + x_4 &= 0 \\
   6x_1 - x_2 + x_4 &= 0
   \end{align*}
   \]
   a. Use Cramer’s rule to solve for $x_1$.
   b. Use Gaussian elimination to solve for $x_1$.

5. Consider the following system of equations:
   
   \[
   \begin{align*}
   -x_1 + 3x_2 - x_3 &= 7 \\
   x_1 - x_2 - x_3 &= -2 \\
   -x_1 + 6x_2 + 2x_3 &= 3
   \end{align*}
   \]
   a. Solve this system using Cramer’s rule.
   b. Solve using Gaussian elimination.
   c. Solve by finding the inverse of the coefficient matrix.

6. Solve the following system by using Cramer’s rule.
   
   \[
   \begin{align*}
   2x_1 - x_3 &= 1 \\
   2x_1 + 4x_2 - x_3 &= 0 \\
   x_1 - 8x_2 - 3x_3 &= -2
   \end{align*}
   \]

7. Solve using Cramer’s rule.
   
   \[
   \begin{align*}
   x_1 + x_2 + x_3 &= a \\
   x_1 + (1 + a)x_2 + x_3 &= 2a \\
   x_1 + x_2 + (1 + a)x_3 &= 0
   \end{align*}
   \]

8. Solve using Cramer’s rule.
   
   \[
   \begin{align*}
   x_1 + x_2 + x_3 + x_4 &= 4 \\
   x_1 + 2x_2 + 2x_3 + 2x_4 &= 1 \\
   x_1 + x_2 + 2x_3 + 2x_4 &= 2 \\
   x_1 + x_2 + x_3 + 2x_4 &= 3
   \end{align*}
   \]
4.4 Area and Volume

Given any two vectors in \( \mathbb{R}^2 \) that are not parallel, they determine a parallelogram; cf. problems 9 and 10 in Section 2.1. With a few manipulations it is possible to express the area of this parallelogram as the absolute values of the determinant of a matrix constructed from the coordinates of these vectors.

**Example 1.** Compute the area of the parallelogram determined by the vectors \((1,2)\) and \((8,0)\) (see Figure 4.2).

The area of this parallelogram is the base length 8 times the height 2.

\[
\text{Area} = 8 \times 2 = \det \begin{bmatrix} 8 & 1 \\ 0 & 2 \end{bmatrix} \tag{4.8}
\]

where the columns of the matrix \( \begin{bmatrix} 8 & 1 \\ 0 & 2 \end{bmatrix} \) are the coordinates of the two vectors \((8,0)\) and \((1,2)\), with respect to the standard basis of \( \mathbb{R}^2 \). □

![Figure 4.2](image)

We derive a similar formula for any two vectors \((a_1, a_2)\) and \((b_1, b_2)\). Our proof and picture assume that both vectors lie in the first quadrant, but the other cases can be handled in a similar manner.

Consider the parallelogram determined by the two vectors \((a_1, a_2)\) and \((b_1, b_2)\) (see Figure 4.3).

\[
\text{Area of parallelogram } OBCA = \text{area of triangle}(OBP_2) \\
+ \text{area of trapezoid}(P_2BCP_3) \\
- \text{area of triangle}(OAP_1) \\
- \text{area of trapezoid}(P_1ACP_3) \\
= \frac{1}{2}(b_1 b_2) + \frac{1}{2}(a_1)(b_2 + a_2 + b_2) \\
- \frac{1}{2}(a_1 a_2) - \frac{1}{2}(b_1)(a_2 + a_2 + b_2) \\
= a_1 b_2 - a_2 b_1 = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}
\]
Since we do not wish to worry about the order in which we list the columns of
the above matrix, we write the area as

\[ \text{Area} = \left| \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right| \]

We formally state this in the following theorem.

**Theorem 4.9.** Let \( \mathbf{a} = (a_1, a_2) \) and \( \mathbf{b} = (b_1, b_2) \) be any two vectors in \( \mathbb{R}^2 \). Let \( P \) be the parallelogram generated by these two vectors, i.e., \( P = \{ \mathbf{x} : \mathbf{x} = t_1 \mathbf{a} + t_2 \mathbf{b}, t_1 \) and \( t_2 \) arbitrary scalars between 0 and 1 \}. Then

\[ \text{Area}(P) = \left| \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right| \]

**Example 2.**

a. Sketch the parallelogram determined by the vectors \((-6, 4)\) and \((3, -2)\) and calculate its areas.

We see from Figure 4.4a that the parallelogram is just the straight-line segment joining the two points \((-6, 4)\) and \((3, -2)\). Thus the area should be zero. Indeed, we have

\[ \text{Area}(P) = \left| \det \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \right| = |12 - 12| = 0 \]

b. Repeat the above, using the vectors \((1, -2)\) and \((-6, -3)\) (see Figure 4.4b).

\[ \text{Area}(P) = \left| \det \begin{bmatrix} 1 & -6 \\ -2 & -3 \end{bmatrix} \right| = |(-3 - 12)| = 15 \]
As one would expect, there is a generalization of this formula to higher dimensions, and we state it in the next theorem.

**Theorem 4.10.** Let \( \{ \mathbf{x}_1, \ldots, \mathbf{x}_n \} \) be any \( n \) vectors in \( \mathbb{R}^n \). Let \( P \) be the \( n \)-dimensional parallelepiped generated by these vectors, that is,

\[
P = \{ \mathbf{y} : \mathbf{y} = t_1 \mathbf{x}_1 + \cdots + t_n \mathbf{x}_n, \ 0 \leq t_j \leq 1 \}
\]

Then the \( n \)-dimensional volume of \( P \) equals

\[
\text{Vol}(P) = \left| \begin{array}{cccc}
\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots
\end{array} \right| 
\]

(4.9)

The matrix in (4.9) is obtained by using the coordinates of the vector \( \mathbf{z}_j \) (with respect to the standard basis) as the \( j \)th column. Thus, if we had the four
vectors
\[ \mathbf{x}_1 = (1, 1, 2, 1) \quad \mathbf{x}_2 = (-1, -1, 3, 4) \quad \mathbf{x}_3 = (8, 9, 1, 1) \quad \mathbf{x}_4 = (10, 11, 1, 0) \]
then the matrix would be
\[
\begin{pmatrix}
1 & -1 & 8 & 10 \\
1 & -1 & 9 & 11 \\
2 & 3 & 1 & 1 \\
1 & 4 & 1 & 0
\end{pmatrix}
\]

**Example 3.** Sketch the parallelepiped generated by the three vectors \( \mathbf{a} = (1, 1, 0), \mathbf{b} = (0, 1, 1), \) and \( \mathbf{c} = (1, 0, 1) \) (see Figure 4.5), and determine its volume.

\[
\text{Vol}(P) = \left| \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \right|
= \left| \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right|
= |1 + 1| = 2
\]

Figure 4.5

We remark that these determinants will be zero if and only if the \( n \) vectors used to form the matrices are linearly dependent. In that case, the solid they generate
will lie in an \((n-1)\)-dimensional plane, and hence should have \(n\)-dimensional volume equal to zero. See Example 2a.

If \(L\) is a linear transformation from \(\mathbb{R}^2\) to \(\mathbb{R}^2\), it has a matrix representation \(A\), where

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

As we saw in Chapter 3, \(L(e_1) = (a_{11}, a_{21})\) and \(L(e_2) = (a_{12}, a_{22})\). Thus, geometrically we can picture \(L\) as transforming the parallelogram generated by \(e_1\) and \(e_2\) into the parallelogram generated by \((a_{11}, a_{21})\) and \((a_{12}, a_{22})\) (see Figure 4.6). Moreover, we have

\[
\text{Area}(L(P)) = \left| \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = |\det(A)|
\]

Since in this particular case we have \(\text{area}(P) = 1\), we may rewrite this formula as

\[
\text{Area}(L(P)) = |\det(A)|\text{area}(P)
\]

To see that this formula is true in general let \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\). Let \(P\) be the parallelogram generated by \(x\) and \(y\), that is,

\[
P = \{t_1x + t_2y: 0 \leq t_1 \leq 1, 0 \leq t_2 \leq 1\}
\]

Then

\[
L(P) = \{t_1L(x) + t_2L(y): 0 \leq t_1 \leq 1, 0 \leq t_2 \leq 1\}
\]

where

\[
L(x) = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)
\]

and

\[
L(y) = (a_{11}y_1 + a_{12}y_2, a_{21}y_1 + a_{22}y_2).
\]
We have from (4.9) that

\[
\text{Area}(L(P)) = |\det \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{11}y_1 + a_{12}y_2 \\ a_{21}x_1 + a_{22}x_2 & a_{21}y_1 + a_{22}y_2 \end{bmatrix}|
\]

\[
= |\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} |\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}| = |\det(A)||\text{area}(P)|
\]

Thus, a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) maps parallelograms into parallelograms (rank \( A = 2 \)), or line segments (rank \( A = 1 \)), or a point (rank \( A = 0 \)). Moreover, the change in area depends only on the linear transformation and not on the particular parallelogram. Naturally there is a generalization of this to higher dimensions which we state below.

**Theorem 4.11.** Let \( A \) be the \( n \times n \) matrix representation of \( L : \mathbb{R}^n \to \mathbb{R}^n \), with respect to the standard basis of \( \mathbb{R}^n \). Let \( P \) be an \( n \)-dimensional parallelogram generated by the \( n \) vectors \( \{x_1, \ldots, x_n\} \). Then \( L(P) \) is generated by the \( n \) vectors \( \{L(x_1), \ldots, L(x_n)\} \) and their volumes are related by the formula

\[
\text{Vol}(L(P)) = |\det(A)||\text{Vol}(P)| \tag{4.10}
\]

**Problem Set 4.4**

1. Calculate the area of the triangles whose vertices are:
   a. \((0,0), (1,6), (-2,3)\)  
   b. \((8,17), (9,2), (4,6)\)

2. Calculate the volume of the tetrahedron whose vertices are:
   a. \((0,0,0), (1, -1, 2), (-3, 6, 7), (1,1,1)\)
   b. \((1,1,1), (-1, -1, -1), (0,4,8), (-3, 0, 2)\)

3. Find the area of the parallelograms determined by the following vectors; cf. problem 1.
   a. \((1,6), (-2,3)\)  
   b. \((1, -15), (-4, -11)\)

4. Sketch the parallelograms \( P \) generated by the following pairs of vectors.
   Let \( O \) be the parallelogram generated by the standard basis vectors. For each of the parallelograms \( P \) find a linear transformation \( L \) such that \( P = L(O) \).
   a. \((1,0), (1,2)\)  
   b. \((1, -1), (3,6)\)
5. In $\mathbb{R}^2$, show that the straight line passing through the two points $(a_1, a_2)$ and $(b_1, b_2)$ has equation
\[
\det \begin{bmatrix} x_1 & x_2 & 1 \\ a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \end{bmatrix} = 0
\]

6. Let $L$, a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, have matrix representation
\[
A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}
\]
Let $P$ be the parallelogram generated by the vectors $(-1, 2)$ and $(1, 3)$. Sketch $P$ and $L(P)$ and compute their areas. Then verify (4.10).

7. Let $A = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$ be the matrix representation of a linear transformation $L$. Let $P$ be the parallelogram generated by the following pairs of vectors:
   a. $(1,0), (0,1)$
   b. $(1,1), (-1,1)$
In each case sketch $P, L(P)$, and verify (4.10).

8. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ be the matrix representation of a linear transformation $L$. Let $P$ be the solid generated by the following vectors:
   a. $(1,0,0), (0,1,0), (0,0,1)$
   b. $(1,1,0), (1,0,1), (0,1,1)$
In each case sketch $P, L(P)$, and verify (4.10).

**Supplementary Problems**

1. Let $A = [a_{jk}]$ be an $n \times n$ matrix. Define each of the following:
   a. $M_{jk}$ minor of $A$
   b. Cofactor of $a_{jk}$
   c. $\text{Adj}(A)$

2. Compute the determinant and adjoint of each of the following matrices:
   a. $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$ $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$
b. \[
\begin{bmatrix}
-2 & 3 & 4 \\
1 & 0 & -2 \\
5 & 1 & 6
\end{bmatrix}
\]
\[
\begin{bmatrix}
5 & -3 & 1 \\
0 & 1 & 1 \\
-5 & 5 & 1
\end{bmatrix}
\]

3. Let \[A = \begin{bmatrix}
-2 & x & 1 \\
x & 1 & 1 \\
2 & 3 & -1
\end{bmatrix}\]. Find all values of \(x\) for which \(\det(A) = 0\).

4. Find all values of \(\lambda\) for which the following system of equations has a nontrivial solution:
\[
\begin{align*}
2x_1 - 7x_2 &= \lambda x_1 \\
-4x_1 + x_2 &= \lambda x_2
\end{align*}
\]

5. Let \(f(t) = \det \begin{bmatrix}
a(t) & b(t) \\
c(t) & d(t)
\end{bmatrix}\), which \(a(t), b(t), c(t),\) and \(d(t)\) are differential functions of \(t\).

   a. Show that \(f'(t) = \det \begin{bmatrix}
a' & b' \\
c & d
\end{bmatrix} + \det \begin{bmatrix}
a & b \\
c' & d'
\end{bmatrix}\).

   b. Derive a similar formula for the determinant of a \(3 \times 3\) matrix of functions.

6. Let \(y_1(t)\) and \(y_2(t)\) be two solutions of the differential equation
\[
ay'' + by' + cy = 0
\]
Define the Wronskian \(W(t)\) of the solutions \(y_j(t)\) by
\[
W(t) = \det \begin{bmatrix}
y_1(t) & y_2(t) \\
y_1'(t) & y_2'(t)
\end{bmatrix}
\]
Show that the Wronskian satisfies the differential equation
\[
aW' + bW = 0
\]

7. Using the fact that a matrix has nonzero determinant if and only if its rows (columns) are linearly independent, determine which of the following sets are linearly independent:

   a. \((2,1), (-3,4)\)

   b. \((2, 1, -2, 3), (0,1,0,1), (1,1,1,0), (3,3,-1,4)\)

8. If \(A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\) and \(B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}\), then the cross product of \(A\) and \(B\) is defined as
\[
A \times B = \det \begin{bmatrix}
i & j & k \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{bmatrix}
\]
\[
= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)
\]
The reader should realize that the term involving the determinant is merely a mnemonic, useful in remembering the last expression. Show that \(A \times B = -B \times A\) and the \(A \times A = 0\).
9. Verify the following formulas:

a. \[ \det \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} e \]

b. \[ \det \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \det \begin{bmatrix} e & f \\ g & h \end{bmatrix} \]

10. Let \( A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \) be an \( n \times n \) matrix, where \( A_1 \) is an \( m \times m \) matrix and \( A_2 \) is an \( (n-m) \times (n-m) \) matrix, \( 1 \leq m \leq n-1 \). Show that

\[ \det(A) = \det(A_1) \det(A_2) \]

11. A group \((G, \cdot)\) is a mathematical system that consists of a collection of objects \( G \) and an operation \( \cdot \) between two elements of \( G \), which gives an element of \( G \). Thus, if \( x \) and \( y \) are elements of \( G \), then \( x \cdot y \) (we drop the dot in the future) is also in \( G \). We also suppose that this operation, called multiplication, satisfies the following properties:

1. \((xy)z = x(yz)\).
2. There is an identity element \( e \) in \( G \) such that \( ex = xe = x \) for every \( x \) in \( G \).
3. For each \( x \) in \( G \), there is an \( x^{-1} \) in \( G \) such that \( xx^{-1} = x^{-1}x = e \)

a. Show that the set of vectors in a vector space forms a group, the group operation being vector addition.

b. Let \( GL(n) \) denote the set of invertible \( n \times n \) matrices, with the group operation being matrix multiplication. Show that \( GL(n) \) forms a group. This group is called the general linear group of order \( n \).

c. Let \( SL(n) \) denote the set of invertible \( n \times n \) matrices with determinant equal to 1. Show that \( SL(n) \), with the group operation being matrix multiplication, is a group. This group is called the special linear group of order \( n \).

d. Show that the set of all \( n \times n \) matrices under matrix multiplication does not form a group.

12. An \( n \times n \) matrix \( P \) is said to be orthogonal if \( P^{-1} = P^T \).

a. Show that if \( P \) is orthogonal, then \( \det(P) = \pm 1 \).

b. Deduce from part a that if \( P \) is orthogonal and if \( K \) is some \( n \)-dimensional parallelepiped, then \( \text{vol}(PK) = \text{vol}(K) \).
c. Find a 2 × 2 matrix $P$ that is not orthogonal, and for which $\det(P)$ equals 1.

d. Show that $O(n)$, the set of $n \times n$ orthogonal matrices, forms a group under the operation of matrix multiplication; cf. problem 11.

13. A matrix $B$ is said to be similar to the matrix $A$ if there is a nonsingular matrix $P$ such that $B = PAP^{-1}$.

a. Show that if $B$ is similar to $A$, then $\det(B) = \det(A)$.

b. Find a pair of $2 \times 2$ matrices $A$ and $B$ for which $\det(A) = \det(B)$ and $B$ is not similar to $A$. Hint: If $B$ is similar to a scalar matrix, then $B = A$.

14. Find two $2 \times 2$ matrices $A$ and $B$ for which $\det(A + B) \neq \det(A) + \det(B)$.