Chapter 2

Vector Spaces

One of my favorite dictionaries (the one from Oxford) defines a vector as “A quantity having direction as well as magnitude, denoted by a line drawn from its original to its final position.” What is useful about this definition is that we can draw pictures and use our spatial intuition. An objection to this being used as the definition of a vector is: It’s not very precise, and thus will be hard to compute with to any degree of accuracy.

The first section of this chapter makes the phrase “a quantity having direction and magnitude” more precise and at the same time develops an algebraic structure. That is, the addition of one vector to another and the multiplication of vectors by numbers will be defined, and various properties of these operations will be stated and proved.

In order to avoid confusing vectors with numbers, all vectors will be in boldface type.

2.1 $\mathbb{R}^2$ through $\mathbb{R}^n$

Fix some point (think of it as the origin in the Euclidean plane), draw two short rays from this point, and put arrowheads at the tips of the rays (see Figure 2.1).

![Figure 2.1](image)

You’ve just drawn two vectors. Now label one of them $A$ and the other $B$. The magnitudes of these vectors are their lengths.

In many applications, vectors are used to represent forces. Since several forces may act on an object, and the result of this will be the same as if a single
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force, called the resultant or sum, acted on the object, we define the sum of
two vectors in order to model how forces combine. Thus, if we want the vector
\( \mathbf{A} + \mathbf{B} \), we draw a dashed line segment starting at the tip of \( \mathbf{A} \), parallel to \( \mathbf{B} \),
with the same length as \( \mathbf{B} \); cf. Figure 2.2a and b. Label the end of this dashed
line \( \mathbf{c} \) and now draw the vector \( \mathbf{C} \); i.e., draw a line segment from the origin to \( \mathbf{c} \)
and put an arrowhead there; cf. Figure 2.2a. The vector \( \mathbf{C} \) is called \( \mathbf{A} + \mathbf{B} \), and
this way of combining \( \mathbf{A} \) and \( \mathbf{B} \) is referred to as the parallelogram law of vector
addition. Note that the construction of \( \mathbf{B} + \mathbf{A} \) will be different from that of
\( \mathbf{A} + \mathbf{B} \); cf. Figure 2.2b. However, a little geometry convinces us that we get the
same two vectors. We repeat this. The fact that \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \) is something
that has to be proved. Its truth is not self-evident.

There is another operation (scalar multiplication) that we can perform on a
vector, and that is to multiply it by a number. If \( \mathbf{A} \) is a vector, \( 2\mathbf{A} \)'s meaning
is clear. It is the same as \( \mathbf{A} + \mathbf{A} \); i.e., \( 2\mathbf{A} \) is a vector twice as long, and with the
same direction, as \( \mathbf{A} \). Thus, we define \( c\mathbf{A} \) (\( c \geq 0 \)) to be the vector pointing in
the same direction as \( \mathbf{A} \) but whose magnitude is \( c \) times the magnitude of \( \mathbf{A} \). If \( c \)
is negative, \( c\mathbf{A} \) points in the direction opposite to \( \mathbf{A} \).

Since we will be doing a lot of computing with vectors, we need a method
for doing so other than using a ruler, compass, and protractor. Following in the
footsteps of Descartes and others, we assign a pair of numbers to each vector and
then see how these pairs should be combined in order to model vector addition
and scalar multiplication.

\[ \begin{align*}
&\quad \mathbf{A}, \quad \mathbf{B}, \quad \mathbf{A} + \mathbf{B} \\
&\quad \mathbf{A}, \quad \mathbf{B}, \quad \mathbf{B} + \mathbf{A}
\end{align*} \]

Figure 2.2

Let’s go back to Figure 2.1. This time we label our initial point \((0,0)\),
the origin in the Euclidean plane. We also draw two perpendicular lines that
intersect at \((0,0)\). We call the horizontal line the \( x_1 \) axis and the vertical line
the \( x_2 \) axis. We next associate a number with each point on these axes. This
number indicates the directed distance of the point from the origin; that is,
the point on the \( x_1 \) axis labeled 2 is 2 units to the right of the origin while
the point labeled \(-2\) is 2 units to the left. How large a distance the number
1 represents is arbitrary. For the \( x_2 \) axis, a positive number indicates that the
point lies above the origin while a negative number means that the point lies
below the origin. We now associate an ordered pair of numbers with each point
in the plane. The first number tells us the directed distance of the point from
the \( x_2 \) axis along a line parallel to the \( x_1 \) axis and the second number gives us
the directed distance of the point from the \( x_1 \) axis along a line parallel to the
Every vector, which we picture as an arrow emanating from the origin, is uniquely determined once we know where its tip is located. This means that every vector can be uniquely associated with an ordered pair of numbers. In other words, two vectors are equal if and only if their respective coordinates are the same.

The preceding discussion may lead to some confusion as to what an ordered pair of numbers represents: a point in the plane or a vector. In practice this will not cause any problem, since one can just as easily think of a vector as a point in the plane, that point where the tip of the vector is located.

**Example 1.** Sketch the following vectors:

a. (1,1)

b. (−1,3)
Our next task is to determine how the number pairs, i.e., the coordinates, of two vectors should be combined to give their sum. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two vectors. Then a simple proof using congruent triangles yields that $A + B = (a_1 + b_1, a_2 + b_2)$; cf. Figure 2.6a. Similar arguments show us that $cA = (ca_1, ca_2)$ if $c$ is a rational number; cf. Figure 2.6b. We haven’t talked about subtracting one vector from another yet, but $A - B$ should be equal to a vector such that $(A - B) + B = A$. Thus if $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then $A - B = (a_1 - b_1, a_2 - b_2)$.

We now formally define $\mathbb{R}^2$ along with vector addition and scalar multiplication for these particular vectors. $\mathbb{R}^2$ is the classical example of a two-dimensional vector space.
2.1. $\mathbb{R}^2$ THROUGH $\mathbb{R}^N$

Definition 2.1. $\mathbb{R}^2 = \{(x_1, x_2) : x_1$ and $x_2$ are arbitrary real numbers$\}$. If $A = (a_1, a_2), B = (b_1, b_2),$ and $c$ is any real number, then

1. $A + B = (a_1 + b_1, a_2 + b_2)$
2. $cA = (ca_1, ca_2)$

Note that $\mathbb{R}^2$ can also be thought of as all possible $1 \times 2$ matrices, with vector addition being matrix addition and scalar multiplication the same as multiplying a matrix by a scalar; cf. Definitions 1.3 and 1.4.

A few words about the notation $\{ : \}$. It is used to describe a set of elements. The phrase after the colon describes the property or attributes that something must possess in order for it to be in the set. For example, $\{x : x = 1, 2\}$ is the set $\{1, 2\}$, and $\{x: 1 \leq x < 100\}$ is the set of real numbers between 1 and 100 including 1 but excluding 100. Thus $(1, 23), (3.14159, 53)$, and $(0, -7)$ are in $\mathbb{R}^2$, but $(0, \$), (\#, f),$ and !e, are not, since the last three objects are not pairs of real numbers.

Example 2. Solve the following vector equation:

$$2(6, -1) + 3X = (3, -4)$$
$$3X = (3, -4) - 2(6, -1)$$
$$= (3, -4) - (12, -2)$$
$$= (3 - 12, -4 + 2) = (-9, -2)$$
Thus

\[ \mathbf{X} = \frac{1}{3}(-9, -2) = \left( -3, -\frac{2}{3} \right) \] \hfill \Box

**Example 3.** Given any vector in \( \mathbb{R}^2 \) write it as the sum of two vectors that are parallel to the coordinate axes.

**Solution.** Let \( \mathbf{A} = (a_1, a_2) \) be any vector in \( \mathbb{R}^2 \). To say that a vector \( \mathbf{X} \) is parallel to the \( x_1 \) axis is to say that the second component of \( \mathbf{X} \) ’s representation as an ordered pair of numbers is zero. A similar comment applies to vectors parallel to the \( x_2 \) axis. Thus,

\[ \mathbf{A} = (a_1, a_2) = (a_1, 0) + (0, a_2) \]

The first vector, \( (a_1, 0) \), is parallel to the \( x_1 \) axis and the second vector, \( (0, a_2) \), is parallel to the \( x_2 \) axis. We could also write

\[ \mathbf{A} = (a_1, 0) + (0, a_2) = a_1(1, 0) + a_2(0, 1) \]

The vectors \((1,0)\) and \((0,1)\) are often denoted by \( \mathbf{i} \) and \( \mathbf{j} \), respectively, and in this notation \((a_1, a_2) = a_1 \mathbf{i} + a_2 \mathbf{j}\).

![Figure 2.7](image-url)

The theorem below lists some of the algebraic properties that vector addition and scalar multiplication in \( \mathbb{R}^2 \) satisfy.

**Theorem 2.1.** Let \( \mathbf{A} = (a_1, a_2) \), \( \mathbf{B} = (b_1, b_2) \), and \( \mathbf{C} = (c_1, c_2) \) be three arbitrary vectors in \( \mathbb{R}^2 \). Let \( a \) and \( b \) be any two numbers. Then the following equations are true.

1. \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \)
2. \( (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \)
3. Let \( \mathbf{0} = (0, 0) \), then \( \mathbf{A} + \mathbf{0} = \mathbf{A} \) [zero vector]
4. For every \( \mathbf{A} \) there is a \( (-\mathbf{A}) \) such that \( \mathbf{A} + (-\mathbf{A}) = \mathbf{0} \) [\( -\mathbf{A} = (-a_1, -a_2) \)]
5. \( a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B} \)
6. \((a + b)A = aA + bA\)
7. \((ab)A = a(bA)\)
8. \(1A = A\)

**Proof.** We verify equations 1, 4, 5, and 8, leaving the others for the reader.

1. \(A + B = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (b_1, b_2) + (a_1, a_2) = B + A\)

2. \(A + (-A) = (a_1, a_2) + (-a_1, -a_2) = (a_1 - a_1, a_2 - a_2) = (0, 0) = 0\)

3. \(a(A + B) = a[(a_1, a_2) + (b_1, b_2)] = a(a_1 + b_1, a_2 + b_2) = (aa_1 + ab_1, aa_2 + ab_2) = (aa_1, aa_2) + (ab_1, ab_2) = a(a_1, a_2) + a(b_1, b_2) = aA + aB\)

4. \(1A = 1(a_1, a_2) = (1a_1, 1a_2) = (a_1, a_2) = A\)

We next discuss the standard three-dimensional space \(\mathbb{R}^3\). As with \(\mathbb{R}^2\), we first picture arrows starting at some fixed point and ending at any other point in three-dimensional space. We draw three mutually perpendicular coordinate axes \(x_1, x_2,\) and \(x_3\) and impose a distance scale on each of the axes. To each point \(P\) in three space we associate an ordered triple of numbers \((a, b, c)\), where \(a\) denotes the directed distance from \(P\) to the \(x_2, x_3\) plane, \(b\) the directed distance from \(P\) to the \(x_1, x_3\) plane, and \(c\) the directed distance from \(P\) to the \(x_1, x_2\) plane. Just as we did for \(\mathbb{R}^2\), we now think of vectors in three space both as arrows and as ordered triples of numbers.

**Example 4.** For each of the triples of numbers below sketch the vector they represent.

a. \((1,0,0), (0,1,0), (0,0,1)\); these three vectors are commonly denoted by \(\mathbf{i}, \mathbf{j},\) and \(\mathbf{k}\), respectively.
This notation is somewhat ambiguous; e.g., does \( i \) represent \((1,0)\) or \((1,0,0)\)? There should, however, be no confusion, since which vector space we are talking about will be clear from the discussion, and this will determine the meaning of \( i \).

b. \( \mathbf{A} = (-1, 2, 1) \)

Definition 2.2 defines \( \mathbb{R}^3 \) and the algebraic operations of vector addition and scalar multiplication for this vector space.

**Definition 2.2.** \( \mathbb{R}^3 = \{ (x_1, x_2, x_3) : x_1, x_2, \text{ and } x_3 \text{ are any real numbers} \} \). Vector addition and scalar multiplication are defined as follows:

1. \( \mathbf{A} \pm \mathbf{B} = (a_1, a_2, a_3) \pm (b_1, b_2, b_3) = (a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3) \), for any vectors \( \mathbf{A} \) and \( \mathbf{B} \) in \( \mathbb{R}^3 \).

2. \( c\mathbf{A} = c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \), for any vector \( \mathbf{A} \) and any number \( c \).

**Example 5.** Let \( \mathbf{A} = (1, -1, 0), \mathbf{B} = (0, 1, 2) \). Compute the following vectors:

a. \( \mathbf{A} + \mathbf{B} = (1, -1, 0) + (0, 1, 2) = (1, 0, 2) \)

b. \( 2\mathbf{A} = 2(1, -1, 0) = (2, -2, 0) \)

Having defined \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), we now define \( \mathbb{R}^n \), i.e., the set of ordered \( n \)-tuples of real numbers.

**Definition 2.3.** \( \mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \text{ are arbitrary real number} \} \). If \( \mathbf{A} \) and \( \mathbf{B} \) are any two vectors in \( \mathbb{R}^n \) and \( a \) is any real number, we define vector addition and scalar multiplication in \( \mathbb{R}^n \) as follows:

1. \( \mathbf{A} \pm \mathbf{B} = (a_1, a_2, \ldots, a_n) \pm (b_1, b_2, \ldots, b_n) = (a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n) \)
2.1. $\mathbb{R}^2$ THROUGH $\mathbb{R}^N$

2. $c\mathbf{A} = c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$

We note that $\mathbb{R}^n$ may also be thought of as the set of $1 \times n$ matrices. Sometimes we will want to think of $\mathbb{R}^n$ as the set of $n \times 1$ matrices, i.e., $\mathbb{R}^n$ consists of columns rather than rows. One reason for this is so that $A\mathbf{x}$ will be defined for $A$ an $m \times n$ matrix and $\mathbf{x}$ in $\mathbb{R}^n$.

There are times when complex numbers are used for scalars. We then define

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n): z_k \text{ are arbitrary complex numbers}\}.$$ Vector addition and scalar multiplication are defined as in $\mathbb{R}^n$, except that the scalars may now be complex numbers instead of just real numbers.

**Example 6.** Write the vector $(-1, 6, 4)$ as a sum of three vectors each of which is parallel to one of the coordinate axes.

$$(-1, 6, 4) = (-1, 0, 0) + (0, 6, 0) + (0, 0, 4)$$

We can also write $(-1, 6, 4)$ as a linear combination of the three vectors $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$.

$$(-1, 6, 4) = -1(0, 1, 0) + 6(0, 1, 0) + 4(0, 0, 1) = -\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}.$$ □

A common mistake is to think that $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$; that is, $W = \{(x_1, x_2, 0): x_1$ and $x_2$ arbitrary$\}$ is equated with $\mathbb{R}^2$. Clearly $W$ is not $\mathbb{R}^2$, since $W$ consists of triples of numbers, while $\mathbb{R}^2$ consists of pairs of numbers.

**Example 7.** Solve the following vector equation in $\mathbb{R}^5$:

$$2(-1, 4, 2, 0, 1) + 6\mathbf{X} = 3(2, 0, 6, 1, -1)$$

$$6\mathbf{X} = (6, 0, 18, 3, -3) - 2(-1, 4, 2, 0, 1) = (8, -8, 14, 3, -5)$$

Thus,

$$\mathbf{X} = \frac{1}{6}(8, -8, 14, 3, -5) = \left(\frac{4}{3}, -\frac{4}{3}, \frac{7}{3}, \frac{1}{2}, -\frac{5}{6}\right).$$ □

The algebraic operations we have defined in $\mathbb{R}^n$ have the same properties as those in $\mathbb{R}^2$, and we list them in the following theorem.

**Theorem 2.2.** Let $\mathbf{A}, \mathbf{B},$ and $\mathbf{C}$ be three arbitrary vectors in $\mathbb{R}^n$. Let $a$ and $b$ be any two numbers. Then the following are true:
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1. \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \)

2. \((\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})\)

3. Let \( \mathbf{0} = (0, 0, \ldots, 0) \), then \( \mathbf{0} + \mathbf{A} = \mathbf{A} \)

4. For every \( \mathbf{A} \) in \( \mathbb{R}^n \) there is a \( (-\mathbf{A}) \) in \( \mathbb{R}^n \) such that \( \mathbf{A} + (-\mathbf{A}) = \mathbf{0} \). If \( \mathbf{A} = (a_1, \ldots, a_n) \), then \( -\mathbf{A} = (-a_1, \ldots, -a_n) \)

5. \( a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B} \)

6. \((a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A} \)

7. \((ab)\mathbf{A} = a(b\mathbf{A}) \)

8. \(1\mathbf{A} = \mathbf{A} \)

We ask the reader to check that these identities are valid.

Problem Set 2.1

1. Sketch the following vectors in \( \mathbb{R}^2 \):
   a. \((-1, 2), (0, 6), -2(1, 4)\)
   b. \((1, 1) + (-1, 1), 3(-2, 1) - 2(1, 1)\)

2. If \( \mathbf{A} = (-1, 6) \) and \( \mathbf{B} = (3, -2) \), sketch the vectors
   a. \( \mathbf{A}, -\mathbf{A} \)
   b. \( \mathbf{A} + \mathbf{B}, -\mathbf{A} + \mathbf{B}, \mathbf{A} - \mathbf{B} \)
   c. \( 2\mathbf{A} - 3\mathbf{B} \)

3. Let \( \mathbf{A} = (1, -1) \). Sketch all vectors of the form \( t\mathbf{A} \), where \( t \) is an arbitrary real number.

4. Let \( \mathbf{A} = (1, -2) \). Let \( \mathbf{B} = (-2, 6) \). Find \( \mathbf{X} \) such that \( 6\mathbf{X} + 2\mathbf{A} = 3\mathbf{B} \).

5. Let \( \mathbf{A} \) and \( \mathbf{B} \) be the same vectors as in problem 4. Sketch vectors of the form \( \mathbf{X} = c_1\mathbf{A} + c_2\mathbf{B} \) for various values of \( c_1 \) and \( c_2 \). Which vectors in \( \mathbb{R}^2 \) can be written in this manner?

6. Prove Theorem 2.2 for \( n = 3 \).

7. Let \( \mathbf{A} = (1, -1, 2), \mathbf{B} = (0, 1, 0) \).
   a. Find all vectors \( \mathbf{x} \) in \( \mathbb{R}^3 \) such that \( \mathbf{x} = t_1\mathbf{A} + t_2\mathbf{B} \) for some constants \( t_1 \) and \( t_2 \).
   b. Is the vector \((0,0,1)\) one of the vectors you found in part a?

8. Let \( \mathbf{x} = (1, 0), \mathbf{y} = (0, 1) \). Give a geometrical description of the following sets of vectors.
2.2. Definition of a Vector Space

In the last section we defined the classical vector spaces \( \mathbb{R}^n \), for \( n = 2, 3, \ldots \). In this section we extract from these sets their crucial properties, and then define a vector space to be anything that has these properties. Thus, let \( V \) denote a set or collection of objects. The elements of \( V \) will be called vectors and we assume that there are two operations defined. The first (vector addition) will be denoted by a “+” sign, and the second (scalar multiplication) by a “·” or by juxtaposition. Specifically if \( \vec{x} \) and \( \vec{y} \) are any two vectors, i.e., elements of \( V \), then \( \vec{x} + \vec{y} \) is in \( V \). If \( a \) is any real number and \( \vec{x} \) any vector, then \( a\vec{x} \) is

\[ \{tx + (1-t)y: \ 0 \leq t \leq 1\} \]
\[ \{t_1\vec{x} + t_2\vec{y}: \ 0 \leq t_1 \leq 1, 0 \leq t_2 \leq 1\} \]
\[ \{tx + (1-t)y: \ \text{any real number}\} \]
\[ \{t_1\vec{x} + t_2\vec{y}: \ 0 \leq t_1, 1 \leq t_2\} \]
\[ \{t_1\vec{x} + t_2\vec{y}: \ t_1 \text{ and } t_2 \text{ arbitrary}\} \]

9. Let \( \vec{x} = (1,2), \vec{y} = (-1,1) \). Describe the following sets of vectors.

a. \( \{tx + (1-t)y: \ 0 \leq t \leq 1\} \)

b. \( \{t_1\vec{x} + t_2\vec{y}: \ 0 \leq t_1 \leq 1, 0 \leq t_2 \leq 1\} \)

c. \( \{tx + (1-t)y: \ \text{any real number}\} \)

d. \( \{t_1\vec{x} + t_2\vec{y}: \ 0 \leq t_1, 1 \leq t_2\} \)

e. \( \{t_1\vec{x} + t_2\vec{y}: \ t_1 \text{ and } t_2 \text{ arbitrary}\} \)

10. Let \( \vec{x} = (1,1,0) \) and let \( \vec{y} = (1,1,1) \). Describe the following sets of vectors:

a. \( \{tx + (1-t)y: \ 0 \leq t \leq 1\} \)

b. \( \{t_1\vec{x} + t_2\vec{y}: \ 0 \leq t_1 \leq 1, 0 \leq t_2 \leq 1\} \)

c. \( \{tx + (1-t)y: \ t \text{ any real number}\} \)

11. Given two sets \( A \) and \( B \), their intersection, \( A \cap B \), is defined as everything they have in common.

\[ A \cap B = \{x: \ x \in A \text{ and } x \in B\} \]

Let

\[ W_1 = \{(x_1, x_2, x_3): \ x_1 = x_2, x_3 = 0\} \]
\[ W_2 = \{(x_1, x_2, x_3): \ x_1 = x_2\} \]
\[ W_3 = \{(x_1, x_2, x_3): \ x_1 + x_2 = 0\} \]

Graph each of the following sets:

a. \( W_1 \cap W_2 \)  b. \( W_2 \cap W_3 \)  c. \( W_1 \cap W_2 \cap W_3 \)

2.2 Definition of a Vector Space

In the last section we defined the classical vector spaces \( \mathbb{R}^n \), for \( n = 2, 3, \ldots \). In this section we extract from these sets their crucial properties, and then define a vector space to be anything that has these properties. Thus, let \( V \) denote a set or collection of objects. The elements of \( V \) will be called vectors and we assume that there are two operations defined. The first (vector addition) will be denoted by a “+” sign, and the second (scalar multiplication) by a “·” or by juxtaposition. Specifically if \( \vec{x} \) and \( \vec{y} \) are any two vectors, i.e., elements of \( V \), then \( \vec{x} + \vec{y} \) is in \( V \). If \( a \) is any real number and \( \vec{x} \) any vector, then \( a\vec{x} \) is
in $V$. The conditions that $x + y$ and $ax$ be in $V$ are expressed by saying that $V$ is closed under vector addition and scalar multiplication. The term “in $V$” has been underlined to underscore the idea that there may be times when an addition or multiplication has been defined such that the result will not be back in $V$. In such cases $V$ is not a vector space.

In order for $V$ to be a vector space, it is not enough that these two operations have been defined. They must also satisfy certain laws: those needed so that a reasonable arithmetic is possible. The requirements are listed in the following definition.

**Definition 2.4.** Let $V$ be a set on which two operations, vector addition and scalar multiplication, have been defined. $V$ will be called a real vector space if the following properties are true:

1. For any $x$ and $y$ in $V$, $x + y$ is in $V$.
2. For any $x$ in $V$, and any real number $a$ in $R$, $ax$ is in $V$.
3. $x + y = y + x$, for any $x$, and $z$ in $V$.
4. $x + (y + z) = (x + y) + z$, for any $x, y$, and $z$ in $V$.
5. There is a vector in $V$ denoted by $0$ such that $x + 0 = x$, for every vector $x$ in $V$.
6. For any vector $x$ in $V$, there is a vector $(−x)$ in $V$ such that $x + (−x) = 0$.
7. $a(x + y) = ax + ay$, for any two vectors $x$ and $y$ in $V$ and any real number $a$.
8. $(a + b)x = ax + bx$, for any two real numbers $a$ and $b$ and any vector $x$.
9. $(ab)x = a(bx)$, for any two real numbers $a$ and $b$ and any vector $x$.
10. $1x = x$, for every vector $x$ in $V$.

What the above properties mean practically is that all the algebraic manipulations you’ve learned to do with real numbers and vectors in $\mathbb{R}^n$ can also be done in this more abstract setting.

Remember, whenever someone says, “$V$ is a real vector space” they are merely stating that the elements of $V$ along with $V$’s version of vector addition and scalar multiplication satisfy rules 1 through 10.

One more comment: The scalars are required to be real numbers and $V$ is called a real vector space. There will be times when the scalars are complex numbers, in which case we call $V$ a complex vector space. In the following we will not explicitly use the adjective real, since all our vector spaces will be real, unless the contrary is stated.
2.2. DEFINITION OF A VECTOR SPACE

Example 1. \( \mathbb{R}^n (n \geq 2) \) is a vector space. Definition 2.3 defined \( \mathbb{R}^n \) as the set of \( n \) tuples of real numbers as well as what vector addition and scalar multiplication mean in this setting. Theorem 2.2 shows that 1 through 10 of Definition 2.4 hold.

Example 2. \( \mathbb{R}^1 = \{ x: x \) is a real number} with vector addition and scalar multiplication being ordinary addition and multiplication of real numbers.

Example 3. \( M_{mn}, \) the set of all \( m \times n \) matrices with real entries, with the usual way of adding matrices and multiplying matrices by numbers, is a vector space.

Example 4. Let \( V = \{(x_1, 0, x_3): \) such that \( x_1 \) and \( x_3 \) are real numbers}. Define addition and scalar multiplication by

\[
    x + y = (x_1, 0, x_3) + (y_1, 0, y_3) = (x_1 + y_1, 0, x_1 + y_3)
\]
\[
    ax = a(x_1, 0, x_3) = (ax_1, 0, x_3)
\]

Clearly \( V \) is closed under these versions of vector addition and scalar multiplication; i.e., 1 and 2 hold. As a matter of fact, one can quickly check that 3 through 7 hold, but 8 does not. For

\[
    ax + bx = a(x_1, 0, x_3) + b(x_1, 0, x_3) = (ax_1, 0, x_3) + (bx_1, 0, x_3) = [(a + b)x_1, 0, 2x_3]
\]

while

\[
    (a + b)x = (a + b)(x_1, 0, x_3) = [(a + b)x_1, 0, x_3]
\]

Since these two expressions will be equal only if \( 2x_3 = x_3 \) or \( x_3 = 0 \) it is clear that 8 is not true for all the elements of \( V \). Thus \( V \) is not a vector space.

Example 5. Let \( V = \{ x: x > 0 \} \). That is, \( V \) is the set of all positive real numbers. In this example \( \oplus \) will denote vector addition and \( \odot \) will denote scalar multiplication. Define these operations as follows:

\[
    x \oplus y = xy; \text{ that is, vector addition will be ordinary multiplication}
\]
\[
    a \odot x = x^a
\]

Since the product of two positive numbers is positive, and a positive number raised to any power is still positive, \( V \) is closed under these two operations; i.e., 1 and 2 in the definition of a vector space hold. We now proceed to check, as we must, each of the other required properties.

3. \( x \oplus y = xy = yx = y \oplus x \)

4. \( x \oplus (y \oplus z) = x(yz) = (xy)z = (x \oplus y) \oplus z \)

5. For the 0 vector we try the number 1: \( 1 \oplus x = 1x = x = x \)
6. For \((-\mathbf{x})\) we try \(x^{-1} \circ x = x^{-1}x = 1 = \mathbf{0}\). Note that if \(x > 0\), so is \(x^{-1}\).

7. \(a \circ (\mathbf{x} \oplus \mathbf{y}) = (xy)^a = (x^a)(y^a) = (a \circ \mathbf{x}) \oplus (a \circ \mathbf{y})\)

8. \((a + b) \circ \mathbf{x} = x^{a+b} = x^a x^b = (a \circ \mathbf{x}) \oplus (b \circ \mathbf{x})\)

9. \((ab) \cdot \mathbf{x} = x^{ab} = (x^b)^a = a \circ (b \circ \mathbf{x})\)

10. \(1 \circ \mathbf{x} = x^1 = \mathbf{x}\)

Notice that in 5, the number 1 is treated as a vector in \(V\), while in 10, it is treated as a scalar. Since \(V\), with these definitions of vector addition and scalar multiplication, satisfies 1 through 10 of Definition 2.4, \(V\) is a vector space.

The moral of this example, if there is one, is that neither “addition” nor “multiplication” need always be what we’re used to. Notice that the zero vector in the above example is not the number zero but the number 1. What is important about the zero vector is not that it be zero (the number) but that it satisfy the algebraic property 5 of Definition 2.4.

**Example 6.** \(V = \{(x_1, x_2, 0): x_1, x_2 \text{ are arbitrary real numbers}\}\). The usual definitions of addition and multiplication of vectors in \(\mathbb{R}^3\) will be used.

The reader is asked to check that all 10 properties do indeed hold and conclude that \(V\) is a vector space.

**Example 7.** \(V = \{f: f\text{ is a real-valued function defined on }[0,1]\}\). For \(f\) and \(g\) in \(V\), \(f + g\) must be in \(V\); that is, it must be a function defined on \([0,1]\).

Thus for each \(t\), \(0 \leq t \leq 1\), define \((f + g)(t) = f(t) + g(t)\). We similarly define \(af\), by \((af)(t) = af(t)\). With these definitions of vector addition and scalar multiplication, 1 and 2 are satisfied. The reader is asked to check that the other eight properties are also true.

**Example 8.** Let \(P_n = \{\text{polynomials in } t \text{ of degree } \leq n\}\) =

\[
\{a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0: \text{ where the } a_j \text{ are real numbers}\}
\]

Thus

\[
P_0 = \{a: a \text{ is any real number}\} = \mathbb{R}^1
\]

\[
P_1 = \{a_0 + a_1 t: a_0 \text{ and } a_1 \text{ are arbitrary real numbers}\}
\]

If \(f = f_0 + \cdots + f_n t^n\) and \(g = g_0 + \cdots + g_n t^n\) are any two polynomials in \(P_n\), we define \((f + g)(t)\) by

\[
(f + g)(t) = (f_0 + g_0) + (f_1 + g_1)t + \cdots + (f_n + g_n)t^n
\]

Thus \(f + g\) is in \(P_n\). We define \(af\) by

\[
af(t) = (af_0) + (af_1)t + \cdots + (af_n)t^n
\]

which is also in \(P_n\). To see that \(P_n\) satisfies the other conditions in Definition 2.4 is routine and is left to the reader. We note that \(P_n\), for each value of \(n\), is a subset of the vector space in Example 7.
2.2. DEFINITION OF A VECTOR SPACE

We remind the reader that a vector space is not just a set, but a set with two operations: vector addition and scalar multiplication. If we change any one of these three, all ten properties must again be checked before we can say that we still have a vector space.

In the future when we deal with any of the standard vector spaces $\mathbb{R}^n$, $P_n$, $M_{mn}$, or the vector space in Example 7, the operations of vector addition and scalar multiplication will not be explicitly stated. The reader, unless told otherwise, should assume the standard operations in these spaces.

Problem Set 2.2

1. Verify the details of Example 6.

2. Let $V = \{(x_1, x_2): x_1$ and $x_2$ are real numbers$\}$. Define addition by
   
   $$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$$
   
   and scalar multiplication by
   
   $$a(x_1, x_2) = (ax_1, ax_2)$$

   Is $V$ a vector space? If not, list all the axioms that $V$ fails to satisfy.

3. Let $V = \{(x_1, x_2): x_1$ and $x_2$ are real numbers$\}$. Define addition and scalar multiplication by
   
   $$(x_1, x_2) + (y_1, y_2) = (x_2 + y_2, x_1 + y_1)$$

   $$a(x_1, x_2) = (ax_1, ax_2)$$

   Is $V$ a vector space? If not, list all the axioms that $V$ fails to satisfy. What happens if we define $a(x_1, x_2) = (ax_2, ax_1)$?

4. Verify that $M_{mn}$ (Example 3) is a vector space.

5. Verify that the set in Example 4 satisfies all the properties of Definition 2.4 except number 8.

6. Let $C[0,1]$ be the set of real-valued continuous functions defined on the interval $[0,1]$. This set is a subset of the vector space defined in Example 7. Define addition and multiplication as in that example, and show that $C[0,1]$ is also a vector space.

7. Show that $V$ in Example 7 is a vector space.

8. Show that $P_n$ for $n = 0, 1, \ldots$, is a vector space.

9. Let $V_0$ be all $2 \times 2$ matrices for which the 1,1 entry is zero. Let $V_1$ be all $2 \times 2$ matrices for which the 1,1 entry is nonzero. Determine if either of these two sets is a vector space. Define addition and multiplication as we did in $M_{mn}$. 
10. Let \( V = \{A, B, \ldots, Z\} \); that is, \( V \) is the set of capital letters. Define \( A + B = C \), \( B + C = D \), \( A + E = F \); i.e., the sum of two letters is the first letter larger than either of the summands, unless \( Z \) is one of the letters to be added. In that case the sum will always be \( A \). Can you define a way to multiply the elements in \( V \) by real numbers in such a way that \( V \) becomes a vector space?

11. Let \( V = \{(x, 0, y): x \) and \( y \) are arbitrary real numbers\}. Define addition and scalar multiplication as follows:
   \[
   (x_1, 0, y_1) + (x_2, 0, y_2) = (x_1 + x_2, y_1 + y_2)
   \]
   \[
   c(x, 0, y) = (cx, cy)
   \]
   Is \( V \) a vector space?

12. Let \( V_c = \{(x_1, x_2): x_1 + 2x_2 = c\} \).
   a. For each value of \( c \) sketch \( V_c \).
   b. For what values of \( c \), if any, is \( V_c \) a vector space?

13. Let \( V_c = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} : a_1 + a_2 + a_3 = c \right\} \). For what values of \( c \) is \( V_c \) a vector space?

14. Let \( V_c = \{p: p \) is in \( P_1 \) and \( p(c) = 0\}\}. For what values of \( c \) is \( V_c \) a vector space?

15. Let \( V = \{A: A \) is in \( M_{23}, and \Sigma_{ij}a_{ij} = 0\}\}. That is, \( A \) is in \( V \) if \( A \) is a \( 2 \times 3 \) matrix whose entries sum to zero. Is \( V \) a vector space?

16. Let \( V_1 = \{(x_1, x_2): x_1^2 + x_2^2 = 1\} \), \( V_2 = \{(x_1, x_2): x_1^2 + x_2^2 \leq 1\} \), \( V_3 = \{(x_1, x_2): x_1 \geq 0, x_2 \geq 0\} \), and \( V_4 = \{(x_1, x_2): x_1 + x_2 \geq 0\} \). Which of these subsets of \( \mathbb{R}^2 \) is a vector space?

17. Let \( V = \{p: p \) is in \( P_{17} \) and \( p(1) + p(6) - p(2) = 0\}\}. Is \( V \) a vector space? Does the subscript 17 have any bearing on the matter? What if the constraint is \( p(1) + p(6) - p(2) = 1\)?

### 2.3 Spanning Sets and Subspaces

The material covered so far has been relatively concrete and easy to absorb, but we now have to start thinking about an abstract concept, vector spaces. The only rules that we may use are those listed in Definition 2.4 and any others we are able to deduce from them. This, especially for the novice, is not easy and is somewhat tedious, but well worth the effort.

In the definition of a vector space, we have as axioms the existence of a zero vector (axiom 5) and an additive inverse (axiom 6) for every vector. One question that occurs is, how many zero vectors and inverses are there? Let’s see
if we can find an answer to this, at least in $\mathbb{R}^2$. Suppose $k = (k_1, k_2)$ is another zero vector in $\mathbb{R}^2$. Thus $x = x + k = (x_1, x_2) + (k_1, k_2) = (x_1 + k_1, x_2 + k_2)$. This implies that $x_1 = x_1 + k_1$ and $x_2 = x_2 + k_2$, from which $k_1 = k_2 = 0$. Hence $k = (0, 0) = 0$, which shows that $\mathbb{R}^2$ has a unique zero vector. Suppose now that $y$ is an additive inverse for $x$ in $\mathbb{R}^2$. Then $0 = x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$. Hence $x_1 + y_1 = 0 = x_2 + y_2$. Thus $y_1 = -x_1$ and $y_2 = -x_2$, and we have uniqueness of inverses, at least in $\mathbb{R}^2$. What about other vector spaces?

**Theorem 2.3.** Let $V$ be a vector space. Then

1. If $0$ and $k$ are both zero vectors (i.e., both satisfy axiom 5), then $0 = k$.
2. If $-x$ and $y$ are both additive inverses of $x$, then $-x = y$.

Thus, in any vector space the zero vector is unique, and the additive inverse $-x$ of any vector $x$ is uniquely determined by $x$.

**Proof.**

1. Since $0$ satisfies 5, we have $k + 0 = k$ but $k$ also satisfies 5; thus $0 + k = 0$. These two equations plus commutativity of addition imply $k = 0$. Thus, there is only one zero vector.

2. $y = y + 0 = y + [x + (-x)] = (y + x) + (-x) = 0 + (-x) = -x$. We remind the reader that $y + x = 0$ since $y$ is assumed to be an additive inverse for $x$.

Some additional properties of vector spaces are listed in the following theorem. The reader should try to prove them for $V$ equal to $\mathbb{R}^2$; then read the proof of Theorem 2.4 and compare his or her version with ours.

**Theorem 2.4.** Let $V$ be a vector space. Let $X$ be any vector in $V$ and $a$ a any number. Then

1. $0X = 0$
2. $a0 = 0$
3. $(-1)X = -X$
4. $aX = 0$ only if $a = 0$ or $X = 0$

**Proof.**

1. Let $x$ be any vector in $V$. Then,

$$x + 0x = 1x + 0x = (1 + 0)x = 1x = x$$

Adding $-x$ to both sides of this equation, we get

$$0 = -x + x = -x + (x + 0x) = (-x + x) + 0x = 0 + 0x = 0x$$
2. If \( a = 0 \), then from 1 above we have \( a0 = 0 \). Thus we may suppose \( a \neq 0 \). Then,
\[
a0 + x = a0 + (aa^{-1})x = a(0 + a^{-1}x) = a(a^{-1}x) = 1x = x
\]
Hence from Theorem 2.3 \( a0 = 0 \).

3. \( x + (-1)x = 1x + (-1)x = [1 + (-1)]x = 0x = 0 \). Theorem 2.3 now implies that \( (-1)x = -x \).

4. Suppose \( ax = 0 \) and \( a \neq 0 \). Then
\[
0 = a^{-1}0 = a^{-1}(ax) = (aa^{-1})x = 1x = x
\]
Thus \( ax = 0 \) only when \( a = 0 \) or \( x = 0 \).

Quite often when the statement of a theorem or definition is read for the first time, its meaning is lost in a maze of strange words and concepts. The reader is strongly urged to always go back to \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) and try to understand what the statement means in this perhaps friendlier setting. For example, before proving Theorem 2.3 we analyzed the special case \( V = \mathbb{R}^2 \).

Given a collection of vectors \( \{x_k : k = 1, \ldots, n\} \), we often wish to form arbitrary sums of these vectors; that is, we wish to look at all vectors \( x \) of the form
\[
x = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{k=1}^{n} c_kx_k
\]  
(2.1)
where the \( c_k \) are arbitrary real numbers. Such sums are called linear combinations of the vectors \( x_k \).

**Example 1.** Let \( x_1 = (1, 2, 0), x_2 = (-1, 1, 0) \). Determine all linear combinations of these two vectors.

**Solution.** Any linear combination of these two vectors must be of the form
\[
x = c_1x_1 + c_2x_2
= c_1(1, 2, 0) + c_2(-1, 1, 0)
= (c_1 - c_2, 2c_1 + c_2, 0)
\]  
Thus, no matter what values the constants \( c_1 \) and \( c_2 \) are, the third component of \( x \) will always be zero, and it seems likely that as \( c_1 \) and \( c_2 \) vary over all pairs of real numbers, so too will the terms \( c_1 - c_2 \) and \( 2c_1 + c_2 \). We conjecture that the set of all linear combinations of these two vectors will be the set of vectors \( S \) in \( \mathbb{R}^3 \) whose third component is zero. To prove this, we only have to show that if \( x \) is in \( S \), then there are constants \( c_1 \) and \( c_2 \) such that \( x = c_1x_1 + c_2x_2 \). Thus suppose \( x \) is in \( S \). Then \( x = (a, b, 0) \) for some numbers \( a \) and \( b \). We want to find constants \( c_1 \) and \( c_2 \) such that
\[
(a, b, 0) = c_1(1, 2, 0) + c_2(-1, 1, 0)
= (c_1 - c_2, 2c_1 + c_2, 0)
\]
or
\[ c_1 - c_2 = a \quad \text{and} \quad 2c_1 + c_2 = b \]

The solution is
\[ c_1 = \frac{a + b}{3} \quad c_2 = \frac{-2a + b}{3} \]

Thus \((a, b, 0)\) is a linear combination of \((1,2,0)\) and \((-1,1,0)\).

The set of all linear combinations of a set of vectors will occur frequently enough that we give it a special name and notation.

**Definition 2.5.** Given a set of vectors \(A = \{x_k: \ k = 1, 2, \ldots, n\}\) in some vector space \(V\), the set of all linear combinations of these vectors will be called their linear span, or span, and denoted by \(S[A]\).

The result of the previous example restated in this notation is

\[
S[(1,2,0),(-1,1,0)] = \{(x_1,x_2,0): \ x_1 \text{ and } x_2 \text{ are arbitrary real numbers}\}.
\]

**Example 2.** Find the span of the vector \((-1,1)\).

\[
S[(-1,1)] = \{c(-1,1): \ c \text{ is any real number}\}
= \{(-c,c): \ c \text{ is any real number}\}
\]

Thus the span of \((-1,1)\) may be thought of as the straight line \(x_1 + x_2 = 0\).

**Example 3.** Find the span of \(A = \{(1,1,0),(-1,0,0)\}\).

\[
S[A] = \{a(1,1,0) + b(-1,0,0): \ a \text{ and } b \text{ arbitrary real numbers}\}
= \{(a - b,a,0): \ a, b \text{ arbitrary numbers}\}
= \{(x,y,0): \ x, y \text{ arbitrary numbers}\}
\]

\[\square\]
The span of a nonempty set of vectors has some properties that we state and prove in the next theorem.

**Theorem 2.5.** Let $S[A]$ be the linear span of the set $A = \{x_k : k = 1, 2, \ldots, n\}$. Then $S[A]$ satisfies axioms 1 through 10 of the definition of a vector space. That is, $S[A]$ is a vector space.

**Proof.**

1. Let $y$ and $z$ be in $S[A]$. Then there are constants $a_j$ and $b_j$, $j = 1, 2, \ldots, n$ such that

$$y = \sum_{j=1}^{n} a_j \mathbf{x}_j \quad z = \sum_{j=1}^{n} b_j \mathbf{x}_j$$

Thus

$$y + z = \sum_{j=1}^{n} a_j \mathbf{x}_j + \sum_{j=1}^{n} b_j \mathbf{x}_j = \sum_{j=1}^{n} (a_j + b_j) \mathbf{x}_j$$

and we see that $y + z$ is also in $S[A]$.

2. Let $\mathbf{x}$ be in $S[A]$ and $k$ a real number. Then there are constants $c_j$, $j = 1, 2, \ldots, n$ such that $\mathbf{x} = \sum_{j=1}^{n} c_j \mathbf{x}_j$. Thus

$$k \mathbf{x} = k \left( \sum_{j=1}^{n} c_j \mathbf{x}_j \right) = \sum_{j=1}^{n} k c_j \mathbf{x}_j$$

and we have that $k \mathbf{x}$ is in $S[A]$.

3. and 4. Commutativity and associativity of vector addition are true because we are in a vector space to begin with.
2.3. SPANNING SETS AND SUBSPACES

5. Clearly each of the constants in a linear combination of the vectors \( x_j \) can be set equal to zero. Thus we have

\[
\sum_{j=1}^{n} 0 x_j = \sum_{j=1}^{n} 0 = 0
\]

and \( S[A] \) contains the zero vector.

6. Let \( x \) be in \( S[A] \). Then \( x = \sum_{j=1}^{n} c_j x_j \) and \( -x = -(\sum_{j=1}^{n} c_j x_j) = \sum_{j=1}^{n} (-c_j) x_j \). Thus, \( -x \) is also in \( S[A] \).

7. 8, 9, and 10 are true because they are true for all the vectors in \( V \), and hence these properties are true for the vectors in \( S[A] \). Remember everything in \( S[A] \) is automatically in \( V \), since \( V \) is a vector space and is closed under linear combinations.

This fact, that the span of a set of vectors is itself a vector space contained in the original vector space, is expressed by saying that the span is a subspace of \( V \).

**Definition 2.6.** Let \( V \) be a vector space. Let \( W \) be a nonempty subset of \( V \). Then if \( W \) (using the same operations of vector addition and scalar multiplication as in \( V \)) is also a vector space, we say that \( W \) is a subspace of \( V \).

**Example 4.** \( V = \mathbb{R}^3 \). Let \( W = \{ (r, 0, 0) : r \text{ is any real number} \} \). Show that \( W \) is a subspace of \( V \).

![Figure 2.10](image)

**Solution.** Since the operations of vector addition and scalar multiplication will be unchanged, we know that \( W \) satisfies properties 3, 4, 7, 8, 9, and 10. Hence, we merely need to verify that \( W \) satisfies 1, 2, 5, and 6 of Definition 2.4. Thus suppose that \( x \) and \( y \) are in \( W \). Then \( x = (r, 0, 0) \) and \( y = (s, 0, 0) \) for some numbers \( r \) and \( s \), and

\[
x + y = (r, 0, 0) + (s, 0, 0) = (r + s, 0, 0)
\]
Thus $x + y$ is in $W$. Now let $a$ be any real number, then
\[ ax = a(r, 0, 0) = (ar, 0, 0) \]
is also in $W$. To see that $0$ and $-x$ are in $W$ we note that
\[ 0 = 0x = (0, 0, 0) \quad \text{and} \quad -x = -(r, 0, 0) = (-r, 0, 0) \]
Thus $W$ is a subspace of $V = \mathbb{R}^3$. \hfill \qedsymbol

**Example 5.** Let $V$ be any vector space and let $A$ be any nonempty subset of $V$. Then $S[A]$ is a subspace of $V$. This is merely Theorem 2.5 restated. \hfill \qedsymbol

**Example 6.** Let $x = (a, b, c)$ be any nonzero vector in $\mathbb{R}^3$. Then $S[x]$, which is a subspace of $\mathbb{R}^3$, is just the straight line passing through the origin and the point with coordinates $(a, b, c)$. See Figure 2.11. The reader should verify the details of this example. \hfill \qedsymbol

To verify that a subset of a vector space is a subspace can be tedious. The following theorem shows that it is sufficient to verify axioms 1 and 2 of Definition 2.4.

![Figure 2.11](image)

**Theorem 2.6.** A nonempty subset $W$ of a vector space $V$ is a subspace of $V$ if and only if

1. For any $x$ and $y$ in $W$, $x + y$ is also in $W$.

2. For any scalar $c$ and any vector $x$ in $W$, $cx$ is in $W$.

**Proof.** If $W$ is a subspace of $V$, clearly 1 and 2 must be true. Thus, suppose 1 and 2 are true. Then we need to verify that axioms 3 through 10 in the definition of a vector space are also true. Since we have not changed how vectors are added or multiplied by scalars, axioms 3, 4, and 7 through 10 are automatically true. Thus, only axioms 5 and 6 need to be verified.
5. To show that zero is in $W$, we use the fact that $W$ is assumed to be nonempty. Let $\mathbf{x}$ be any vector in $W$. Then since $W$ is closed under scalar multiplication we have $\mathbf{0} = 0\mathbf{x}$ and $\mathbf{0}$ must be in $W$.

6. Let $\mathbf{x}$ be any vector in $W$, then $(-1)\mathbf{x} = -\mathbf{x}$ must also be in $W$. $\square$

In the following, whenever we wish to indicate that a subspace $W$ equals $S[A]$, for some set $A$, we will say $W$ is spanned by $A$ or $A$ is a spanning set of $W$.

**Example 7.** Let $W$ be that subspace of $\mathbb{R}^4$ spanned by the vectors $(1, -1, 0, 2)$, $(3, 0, 1, 6)$, and $(0, 1, 1, -1)$. Is the vector $\mathbf{x} = (2, -3, 4, 0)$ in $W$?

**Solution.** If $\mathbf{x}$ is in $W$, it must be a linear combination of the three vectors whose span is $W$. Thus, there should exist constants $c_1, c_2,$ and $c_3$ such that

$$(2, -3, 4, 0) = c_1(1, -1, 0, 2) + c_2(3, 0, 1, 6) + c_3(0, 1, 1, -1)$$

This vector equation has a solution if and only if the following system has a solution.

$$
\begin{align*}
-3c_2 &= 2 \\
-c_1 + c_3 &= -3 \\
c_2 + c_3 &= 4 \\
2c_1 + 6c_2 - c_3 &= 0
\end{align*}
$$

A few row operations on the augmented matrix of this system show that it is row equivalent to the matrix

$$
\begin{bmatrix}
1 & 3 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & -5
\end{bmatrix}
$$

The nonhomogeneous system, which has the above matrix as its augmented matrix, has no solution. This means that $W$ does not contain the given vector. $\square$

**Example 8.** Let $V = \mathbb{R}^2$. Let $W = \{(x, \sin x) : x$ is a real number$\}$. Show that while $W$ contains $\mathbf{0}$ and the additive inverse $-\mathbf{x}$ of every vector $\mathbf{x}$ in $W$, it still is not a subspace.

**Solution.** To see that $\mathbf{0} = (0, 0)$ is in $W$, we only need observe that $\sin 0 = 0$. Thus $(0, \sin 0) = (0, 0)$ is in $W$. If $\mathbf{x}$ is in $W$, then $\mathbf{x} = (r, \sin r)$ for some number $r$. But then

$$-\mathbf{x} = -(r, \sin r) = (-r, -\sin r) = (-r, \sin(-r))$$

must also be in $W$. To see that $W$ is not a subspace, we note that if $W$ were a subspace, $c(1, \sin 1)$ must be in $W$ for any choice of the scalar $c$. Since $\sin 1 > 0$, by making $c$ large enough we will have $c\sin 1 > 1$. Since $-1 \leq \sin x \leq 1$ for all $x$, we cannot have $\sin x = c\sin 1$ for any $x$. Thus, $c(1, \sin 1) = (c, \sin c\sin 1)$ is not in $W$. Hence, $W$ is not closed under scalar multiplication, and it cannot be a subspace. $\square$
Example 9. Let \( A \) be an \( m \times n \) matrix. Let \( K = \{ \mathbf{x} : \mathbf{x} \) is in \( \mathbb{R}^n \) and \( A\mathbf{x} = \mathbf{0} \} \). Thus \( K \) is the solution set of the system of linear homogeneous equations whose coefficient matrix is \( A \). Show that \( K \) is a subspace of \( \mathbb{R}^n \). Notice that \( \mathbb{R}^n \) is being thought of as the set \( n \times 1 \) matrices.

Solution. We first note that \( \mathbf{0} \) is in \( K \), since \( A\mathbf{0} = \mathbf{0} \). Thus \( K \) is nonempty. Suppose now that \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are in \( K \). Then

\[
A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}
\]

and \( K \) is closed under addition. To see that \( K \) is closed under scalar multiplication we have, if \( \mathbf{x} \) is in \( K \) and \( c \) is any number:

\[
A(c\mathbf{x}) = cA(\mathbf{x}) = c\mathbf{0} = \mathbf{0}
\]

Hence, by Theorem 2.6, \( K \) is a subspace of \( \mathbb{R}^n \).

Example 10. Let \( V = \mathbb{R}^4 \) and let \( W = \{ (x_1, x_2, x_3, x_4) : 2x_1 - x_2 + x_3 = 0 \} \). Show \( W \) is a subspace of \( \mathbb{R}^4 \).

Solution. Note that \( W \) is the solution set of a system of homogeneous equations. Thus, by the previous example \( W \) is a subspace.

Example 11. Show that \( F = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \) spans \( M_{22} \).

Solution. Let \( A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \) be an arbitrary vector (matrix) in \( M_{22} \). Then we have to find constants \( c_1, c_2, c_3, \) and \( c_4 \) such that

\[
\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

Thus, we have the equations

\[
a_1 = c_1 + c_2 \quad a_2 = -c_1 \quad a_3 = c_3 + c_4 \quad a_4 = c_3
\]
2.3. SPANNING SETS AND SUBSPACES

Hence,

\[
\begin{bmatrix}
a_1 & a_2 \\
a_3 & a_4
\end{bmatrix} = (-a_2) \begin{bmatrix}
1 & -1 \\
0 & 0
\end{bmatrix} + (a_1 + a_2) \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
+ a_4 \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix} + (a_3 - a_4) \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\]

Thus, \( M_{2,2} = S[F] \). \( \square \)

Problem Set 2.3

1. Show that \( S[(1,0)(0,1)] = \mathbb{R}^2 \).

2. Show that \( S[(1,0,0),(0,1,0)] \) equals the \( x_1, x_2 \) plane in \( \mathbb{R}^3 \).

3. Show that \( S[(1,0,1)(1,2,-1)] = \{ (x_1,x_2,x_3) : x_1 - x_2 - x_3 = 0 \} \).

4. Show that \( S[(1,1)] = \{ (x_1,x_2) : x_1 - x_2 = 0 \} \).

5. Let \( A = \{ (1,-1,0),(-1,0,1) \} \). Determine which if any of the following vectors is in \( S[A] \): \( (1,1,0),(-1,0,1), (2,3,-1) \).

6. Let \( A = \{ (1,2),(6,3) \} \). Show that \( S[A] = \mathbb{R}^2 \).

7. Let \( x = (x_1,x_2) \) be any nonzero vector in \( \mathbb{R}^2 \). Show that \( S[x] \) is a straight line passing through the origin and the point with coordinates \( (x_1,x_2) \).

8. What is \( S[(0,0)] \)?

9. Let \( V \) be a vector space. Let \( W \) be that subset of \( V \) which consists of just the zero vector. Prove the following:
   a. \( W \) is a subspace of \( V \).
   b. Show that \( V \) is a subspace of itself.

10. Let \( A \) be any set of vectors in some vector space \( V \). Show that \( A \) is contained in \( S[A] \). Show that if \( H \) is any subspace of \( V \) containing \( A \), then \( S[A] \) is also in \( H \). Thus, \( S[A] \) is the smallest subspace of \( V \) which contains \( A \).

11. Let \( V = \mathbb{R}^3 \). Let \( A_1 = \{ (1,1,0) \}, A_2 = \{ (1,1,0), (0,0,1) \} \). Show that \( S[A_1] \) is contained in \( S[A_2] \) and describe these two subspaces geometrically.

12. Let \( f(x) = ax \) for some constant \( a \). Show that the graph of \( f = \{ (x,f(x)) : x \) is any real number \} is a subspace of \( \mathbb{R}^2 \). Conversely suppose we have a function \( f(x) \) whose graph is a subspace of \( \mathbb{R}^2 \). Show that \( f(x) = ax \) for some constant \( a \). (Hint: What must \( a \) equal?)
13. Let \( A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 4 & 6 \end{bmatrix} \). Let \( K \) be the solution set of \( Ax = 0 \), for \( x \) in \( \mathbb{R}^3 \).

From Example 9 we know that \( K \) is a subspace of \( \mathbb{R}^3 \). Find a vector \( x_0 \) such that \( S[x_0] = K \).

14. Let \( W = \{ (x_1, x_2, x_3) : x_1 - x_2 + x_3 = c \} \). For which values of \( c \), if any, is \( W \) a subspace of \( \mathbb{R}^3 \)?

15. Example 11 exhibited a spanning set of \( M_{22} \) with four vectors. Can you find a spanning set of \( M_{22} \) with five vectors? With three vectors?

16. Find spanning sets for each of the following vector spaces:
   a. \( V = \{ (x_1, x_2, x_3) : x_1 + x_2 - 6x_3 = 0 \} \)
   b. \( V = \{ p : p \) is in \( P_2 \) and \( p(1) = 0 \} \)
   c. \( V = \{ A = [a_{ij}] : A \) is in \( M_{23} \) and \( \sum_{j=1}^{3} a_{ij} = 0 \) for \( i = 1, 2 \} \)

17. Let \( V \) be the vector space of all \( n \times n \) matrices. Show that the following subsets of \( V \) are subspaces:
   a. \( W = \{ cI_n : c \) is any number \} \)
   b. \( W = \{ d_j \delta_{jk} : d_j \) any number \} \)
   c. \( W = \{ a_{ij} \) upper triangular matrices \)
   d. \( W = \{ a_{ij} \) lower triangular matrices \)

18. Let \( W \) be the set of all \( n \times n \) invertible matrices. Show that \( W \) is not a subspace of the vector space of all \( n \times n \) matrices.

19. Let \( W \) be any subspace of the vector space \( M_{mn} \) of all \( m \times n \) matrices. Show that \( W^T \), that subset of \( M_{nm} \) consisting of the transposes of all the matrices in \( W \), is a subspace of \( M_{nm} \).

20. Let \( P_2 \) be all polynomials of degree 2 or less. Let \( W = \{ a + bt : a \) and \( b \) arbitrary real numbers \}. Is \( W \) a subspace of \( P_2 \)?

21. Let \( V \) be a vector space. Let \( W_1 \) and \( W_2 \) be any two subspaces of \( V \). Let \( W = W_1 \cap W_2 = \{ x : x \) belongs to both \( W_1 \) and \( W_2 \} \). Show that \( W \) is a subspace of \( V \).

22. Let \( V, W_1, \) and \( W_2 \) be as in problem 21. Let \( W = W_1 \cup W_2 = \{ x : x \) belongs to \( W_1 \) or \( W_2 \} \). Show that \( W \) need not be a subspace of \( V \).

23. Let \( V, W_1, \) and \( W_2 \) be as in problem 21. Define \( W_1 + W_2 \) as that subset of \( V \) which consists of all possible sums of the vectors in \( W_1 \) and \( W_2 \), that is \( W_1 + W_2 = \{ u + v : u \) and \( v \) any vectors in \( W_1 \) and \( W_2 \} \).

Show \( W_1 + W_2 \) is a subspace of \( V \) and that any other subspace of \( V \) which contains \( W_1 \) and \( W_2 \) must also contain \( W_1 + W_2 \). Thus, \( W_1 + W_2 \) is the smallest subspace of \( V \) containing \( W_1 \cup W_2 \).
2.4. LINEAR INDEPENDENCE

In this section we discuss what it means to say that a set of vectors is linearly independent. We encourage the reader to go over this material several times. The ideas discussed here are important, but hard to digest, thus needing rumination.

Consider the two vectors \( \mathbf{x} = (1, 0, -1) \) and \( \mathbf{y} = (0, 1, 1) \) in \( \mathbb{R}^3 \). They have the following property: Any vector \( \mathbf{z} \) in their span can be written in only one way as a linear combination of these two vectors. Let’s check the details of this. Suppose there are constants \( c_1, c_2, c_3, \) and \( c_4 \) such that

\[
\mathbf{z} = c_1 \mathbf{x} + c_2 \mathbf{y} = c_3 \mathbf{x} + c_4 \mathbf{y}
\]  

(2.2)

What we want to show is that \( c_1 = c_3 \) and \( c_2 = c_4 \); that is, the coefficients of \( \mathbf{z} \) must be equal and the coefficients of \( \mathbf{y} \) must also be equal. The two expressions for \( \mathbf{z} \) lead to the equation

\[
0 = \mathbf{z} - \mathbf{z} = (c_1 - c_3) \mathbf{x} + (c_2 - c_4) \mathbf{y}
\]  

(2.3)
Thus, we need to show that anytime $\mathbf{0} = a\mathbf{x} + b\mathbf{y}$ we must have $a = 0 = b$. Suppose now that
\[ \mathbf{0} = (0, 0, 0) = a(1, 0, -1) + b(0, 1, 1) = (a, b, -a + b) \]
Clearly this can occur only if $a = 0 = b$. This property of the two vectors $\mathbf{x}$ and $\mathbf{y}$ is given a special name.

**Definition 2.7.** A set of vectors $\{\mathbf{x}_k : k = 1, \ldots, n\}$ in a vector space $V$ is said to be linearly independent if whenever
\[ \mathbf{0} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \sum_{k=1}^{n} c_k\mathbf{x}_k \]
then
\[ c_1 = c_2 = \cdots = c_n = 0 \]
That is, the zero vector can be written in only one way as a linear combination of these vectors.

A set of vectors is said to be *linearly dependent* if it is not linearly independent. That is, a set of vectors $\{\mathbf{x}_k : k = 1, \ldots, n\}$ is linearly dependent only if there are constants $c_1, c_2, \ldots, c_n$ not all zero such that $\mathbf{0} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$.

**Example 1.** Show that the vectors $\{(1, 0, 1), (2, 0, 3), (0, 0, 1)\}$ are linearly dependent.

*Solution.* We need to find three constants $c_1, c_2, c_3$ not all zero such that
\[ c_1(1, 0, 1) + c_2(2, 0, 3) + c_3(0, 0, 1) = (0, 0, 0) \]
Thus we have to find a nontrivial solution to the following system of equations:
\[ \begin{align*}
    c_1 + 2c_2 &= 0 \\
    c_1 + 3c_2 + c_3 &= 0
\end{align*} \]
This system has many nontrivial solutions. A particular one is $c_1 = 2$, $c_2 = -1$, and $c_3 = 1$. Since we have found a nontrivial linear combination of these vectors, namely, $2(1, 0, 1) - (2, 0, 3) + (0, 0, 1)$, which equals zero, they are linearly dependent.

**Example 2.** Determine whether the set
\[ \{(1, 0, -1, 1, 2), (0, 1, 1, 2, 3), (1, 1, 0, 1, 0)\} \]
is linearly dependent or independent.

*Solution.* We have to decide whether or not there are constants $c_1, c_2, c_3$ not all zero such that
\[ (0, 0, 0, 0, 0) = c_1(1, 0, -1, 1, 2) + c_2(0, 1, 1, 2, 3) + c_3(1, 1, 0, 1, 0) \]
The linear system that arises from this is
\[
\begin{align*}
    c_1 + c_3 &= 0 \\
    c_2 + c_3 &= 0 \\
    -c_1 + c_2 &= 0 \\
    c_1 + 2c_2 + c_3 &= 0 \\
    2c_1 + 3c_2 &= 0
\end{align*}
\]

The only solution is the trivial one, \( c_1 = c_2 = c_3 = 0 \). Thus the vectors are linearly independent. \( \Box \)

To say that a set of vectors is linearly independent is to say that the zero vector can be written in only one way as a linear combination of these vectors; that is, every coefficient \( c_k \) must equal zero. The following theorem shows that even more is true for a linearly independent set of vectors.

**Theorem 2.7.** If \( A = \{x_k: k = 1, \ldots, p\} \) is a linearly independent set of vectors, then every vector in \( S[A] \) can be written in only one way as a linear combination of the vectors in \( A \).

**Proof.** Let \( y \) be any vector in \( S[A] \). Suppose that we have
\[
y = \sum_{k=1}^{p} a_k x_k = \sum_{k=1}^{p} b_k x_k
\]

We wish to show that \( a_k = b_k \) for each \( k \). Subtracting \( y \) from itself we have
\[
0 = y - y = \sum_{k=1}^{p} a_k x_k - \sum_{k=1}^{p} b_k x_k = \sum_{k=1}^{p} (a_k - b_k)x_k
\]

Since the set \( A \) is linearly independent we must have
\[
a_1 - b_1 = 0 \quad a_2 - b_2 = 0 \quad \cdots \quad a_p - b_p = 0
\]

Thus, we can write any vector in \( S[A] \) as a linear combination of the linearly independent vectors \( x_k \) in one and only one way. \( \Box \)

The unique constants \( a_k \) that are associated with any vector \( x \) in \( S[A] \) are given a special name.

**Definition 2.8.** Given a linearly independent set of vectors \( A = \{x_k: k = 1, \ldots, n\} \). If \( x = \sum_{k=1}^{n} a_k x_k \), the coefficients \( a_k \), \( 1 \leq k \leq n \), are called the coordinates of \( x \) with respect to \( A \). Note that there is an explicit ordering of the vectors in the set \( A \).

**Example 3.** Show that the set \( A = \{(1, 1, -1), (0, 1, 0), (-1, 0, 2)\} \) is linearly independent, and then find the coordinates of \( (2, -4, 6) \) with respect to \( A \).

**Solution.** To see that \( A \) is linearly independent suppose
\[
c_1(1, 1, -1) + c_2(0, 1, 0) + c_3(-1, 0, 2) = (0, 0, 0)
\]
CHAPTER 2. VECTOR SPACES

The system of equations that is equivalent to this vector equation, namely,
\[
\begin{align*}
c_1 - c_3 &= 0 \\
c_1 + c_2 &= 0 \\
-c_1 + 2c_3 &= 0
\end{align*}
\]
has only the trivial solution. Thus \( A \) is a linearly independent set of vectors. We next want to write, if possible, \((2, -4, 6)\) as a linear combination of the vectors in \( A \). Thus we want constants \( c_k, k = 1, 2, 3 \), such that
\[
(2, -4, 6) = c_1(1, 1, -1) + c_2(0, 1, 0) + c_3(-1, 0, 2)
\]
The associate system is
\[
\begin{align*}
c_1 - c_3 &= 2 \\
c_1 + c_2 &= -4 \\
-c_1 + 2c_3 &= 6
\end{align*}
\]
and its unique solution is \( c_1 = 10, c_2 = -14, c_3 = 8 \). Thus \((2, -4, 6) = 10(1, 1, -1) - 14(0, 1, 0) + 8(-1, 0, 2)\) and the coordinates of \( z = (2, -4, 6) \) with respect to \( A \) are \([10, -14, 8]\). \( \square \)

Example 4. Show that the set \( F \) is a linearly independent subset of \( M_{22} \).

\[
F = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}
\]

Solution. Suppose there are constants \( c_1, \ldots, c_4 \) such that
\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]
Thus, since the equations
\[
c_1 + c_2 = 0 \quad -c_1 = 0 \quad c_3 + c_4 = 0 \quad c_3 = 0
\]
have only the trivial solution, \( F \) is linearly independent. From Example 11 in Section 2.3 we know that \( S[F] = M_{22} \), and that
\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]
Thus, the coordinates of \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) with respect to \( F \) are \([0, 1, 0, 0]\). \( \square \)

Example 5. Show that the set \( F = \{1 + t, t^2, t - 2\} \) is a linearly independent subset of \( P_2 \).

Solution. Suppose there are constants \( c_1, c_2, \) and \( c_3 \) such that
\[
0 = c_1(1 + t) + c_2(t^2) + c_3(t - 2)
\]
\[
= c_2t^2 + (c_1 + c_3)t + (c_1 - 2c_3)
\]
Then these constants satisfy the equations
\[ c_1 - 2c_3 = 0 \quad c_1 + c_3 = 0 \quad c_2 = 0 \]
Since the only solution to this system is the trivial one, \( F \) is a linearly independent subset of \( P_2 \). \( \square \)

**Theorem 2.8.** A set of vectors \( \{x_k : k = 1, \ldots, n\} \) is linearly dependent if and only if one of its vectors can be written as a linear combination of the remaining \( n - 1 \) vectors.

**Proof.** Suppose one of the vectors, say \( x_1 \), can be written as a linear combination of the others. Then we have
\[ x_1 = c_2 x_2 + c_3 x_3 + \cdots + c_n x_n \]
for some constants \( c_k \). But then
\[ 0 = x_1 - c_2 x_2 - \cdots - c_n x_n \]
and we have found a nontrivial (the coefficient of \( x_1 \) is 1) linear combination of these \( n \) vectors that equals the zero vector. Thus, this set of vectors is linearly dependent. Suppose now that there is a nontrivial linear combination of these vectors that equals 0; that is
\[ 0 = c_1 x_1 + \cdots + c_n x_n \]
and not all the constants \( c_k \) are zero. We may suppose without loss of generality that \( c_1 \) is not zero (merely rename the vectors). Solving the above equation for \( x_1 \), we have
\[ x_1 = \frac{1}{c_1}(-c_2 x_2 - \cdots - c_n x_n) = -\frac{c_2}{c_1} x_2 - \cdots - \frac{c_n}{c_1} x_n. \]
Hence, we have written one of the vectors in our linearly dependent set as a linear combination of the remaining vectors. \( \square \)

If a linearly dependent set contains exactly two vectors, then one of them must be a multiple of the other. In \( \mathbb{R}^3 \), three nonzero vectors are linearly dependent if and only if they are coplanar, i.e., one of them lies in the plane spanned by the other two. For example, the three vectors \( (1,0,1) \), \( (2,0,3) \), and \( (0,0,1) \), as we saw in Example 1, are linearly dependent and all three of them lie in the plane generated by \( (1,0,1) \) and \( (2,0,3) \); that is, the plane \( x_2 = 0 \).

**Problem Set 2.4**

1. Show that \( (1,0,1) \) is in the span of \( \{(2,0,0), (0,0,-1)\} \).
2. Show that \( S[(2,0,0), (0,0,-1)] = S[(2,0,0), (0,0,-1), (1,0,1)] \).

4. Determine whether the following sets of vectors are linearly independent.
   a. $\{(1, 3), (1, -4), (8, 12)\}$
   b. $\{(1, 1, 4), (8, -3, 2), (0, 2, 1)\}$
   c. $\{(1, 1, 2, -3), (-2, 3, 0, 4), (8, -7, 4, -18)\}$

5. Which of the following sets of vectors is linearly dependent?
   a. $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$
   b. $\{(1, -1, 6), (4, 2, 0), (1, 1, 1), (-1, 0, -1)\}$
   c. $\{(2, 1, 4), (1, 1, 1), (-1, 1, 1)\}$

6. Show that any set of vectors containing the zero vector must be linearly dependent.

7. Suppose that $A$ is a linearly dependent set of vectors and $B$ is any set containing $A$. Show that $B$ must also be linearly dependent.

8. Show that any nonempty subset of a linearly independent set is linearly independent.

9. Show that any set of four or more vectors in $\mathbb{R}^3$ must be linearly dependent.

10. Let $A = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$. Is $A$ linearly dependent? $S[A] = ?$

11. Let $A = \{1 + t, 1 - t^2, t^2\}$. Is $A$ a linearly independent subset of $P_2$? $S[A] = ?$

12. Let $A = \{\sin t, \cos t, \sin(t + \pi)\}$. Is $A$ a linearly independent subset of $C[0, 1]$? Does $A$ span $C[0, 1]$?

13. Let $A = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$. Is $A$ a linearly independent subset of $\mathbb{R}^2$? Is $A$ a spanning subset of $\mathbb{R}^2$?

14. Show that $F = \{1 + t, t^2 - t, 2\}$ is a linearly independent subset of $P_2$. Show $S[F] = P_2$.

15. Show that $\{\sin t, \sin 2t, \cos t\}$ is a linearly independent subset of $C[0, 1]$. Does it span $C[0, 1]$?

16. Let $V = \{p(t) : p$ is in $P_2$ and $\int_0^1 p(t)dt = 0\}$. Find a spanning set of $V$ that has exactly two vectors in it. Show that your set is also linearly independent.

17. Let $V = \{p(t) : p$ is in $P_3$ and $p'(0) = 0, p(1) - p(0) = 0\}$. Find a spanning set of $V$ that has exactly two vectors. Is the set also linearly independent?
2.5 Bases

In the preceding two sections we discussed two different types of subsets of a vector space $V$. In Section 2.3 we covered spanning sets, i.e., subsets $A$ with the property that any vector in $V$ can be written as a linear combination of vectors in $A$. Section 2.4 was concerned with sets that are linearly independent. This section discusses properties of sets that are both linearly independent and spanning. Such subsets are called bases of $V$. Besides their defining properties all bases of a vector space share one other common feature: They have exactly the same number of vectors in them. That is, if $A = \{x_1, \ldots, x_p\}$ and $B = \{y_1, \ldots, y_n\}$ are linearly independent and they both span $V$, then $n = p$. Thus the number of vectors in a basis of $V$ depends only on $V$ and not on which linearly independent spanning set we happen to have chosen.

Before proving this useful and interesting fact, we prove two lemmas. These lemmas aid us in the construction of bases.

18. Let $V = \{A: A$ is in $M_{23}$ and $AB = AC = 0_{22}\}$, where

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Find a linearly independent spanning set of $V$. How many vectors are in your set?

19. Let $A$ be a subset of a vector space $V$. We say that $A$ is linearly independent if, whenever $\{x_1, \ldots, x_p\}$ is a finite subset of $A$ and $c_1x_1 + \cdots + c_px_p = 0$, then each $c_k = 0$.

a. Show that if $A$ is a finite set of vectors to begin with, then Definition 2.7 and the above definition are equivalent.

b. Show Theorem 2.7 is also true using this definition of linear independence. Note: $S[A]$ is the collection of all possible finite linear combinations of the vectors in $A$.

20. Let $V = C[0,1]$, the space of real-valued continuous functions defined on $[0,1]$. Let $A = \{1, t, \ldots, t^n, \ldots\}$.

a. Show that $A$ is a linearly independent set as defined in problem 19.

b. What is $S[A]$?

21. Let $V$ be the set of all polynomials with only even powers of $t$ (constants have even degree). Thus $1+t$ is not in $V$, $1+t+t^2$ is not in $V$, but $1+t^2$ and $t^4+5t^8$ are in $V$. Let $A = \{1, t^2, t^4, \ldots, t^{2n}, \ldots\}$.

a. Show that $V$ is a vector space and that $A$ is a linearly independent subset of $V$.

b. $S[A] =$?
Lemma 2.1. Suppose $A = \{x_1, \ldots, x_n\}$ is linearly dependent. Then it is possible to remove one of the vectors $x_k$ from $A$ without changing $S[A]$; that is, $S[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n] = S[A]$.

Proof. Since $A$ is linearly dependent, there are constants $c_k$, $1 \leq k \leq n$, not all zero, such that

$$0 = c_1x_1 + \cdots + c_nx_n \quad (2.4)$$

Suppose, for convenience, that $c_1 \neq 0$. Then we may solve (2.4) for $x_1$ and obtain

$$x_1 = -\frac{1}{c_1} (c_2x_2 + \cdots + c_nx_n) \quad (2.5)$$

We now claim that $S[A] = S[x_2, \ldots, x_n]$. Let $B = \{x_2, \ldots, x_n\}$. Since $B$ is a subset of $A$, we have $S[B] \subseteq S[A]$. Thus, to show these two subspaces of $V$ are equal, we only need show that any vector in $S[A]$ is also in $S[B]$. To this end, let $y$ be in $S[A]$. Then there are constants $a_k$, $1 \leq k \leq n$, such that

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$= a_1 \left( -\frac{1}{c_1} [c_2x_2 + \cdots + c_nx_n] + a_2x_2 + \cdots + a_nx_n \right)$$

$$= \left( a_2 - \frac{a_1c_2}{c_1} \right) x_2 + \cdots + \left( a_n - \frac{a_1c_n}{c_1} \right) x_n$$

Since $y$ can be written as a linear combination of the vectors in $B$, $y$ is in $S[B]$. □

Example 1. Let $A = \{(1,1),(3,2),(1,-1)\}$. It’s easy to see that the span of $A$ is $\mathbb{R}^2$. Moreover since

$$(0,0) = 5(1,1) - 2(3,2) + (1,-1)$$

the set $A$ is linearly dependent. Thus we may remove one of the three vectors from $A$ and still have a set with the same span, i.e., $\mathbb{R}^2$. Which vector can we delete? Since each of the vectors in the linear combination above has a nonzero coefficient, we may delete any one of the three. Let’s remove the second one. Then we know that

$$S[(1,1),(1,-1)] = \mathbb{R}^2$$

We note that these last two vectors are linearly independent. In fact this is one way in which a linearly independent spanning set of a vector space can be constructed. That is, from a linearly dependent spanning set remove, one by one, those vectors that depend linearly on the remaining vectors, until a linearly independent set is obtained. □

Lemma 2.2. Let $A = \{x_1, \ldots, x_n\}$ be a linearly independent set. Suppose that the vector $y$ is not in $S[A]$. Then the set obtained by adding $y$ to $A$ is linearly independent.
2.5. BASES

Proof. Suppose there are constants \( c, c_1, \ldots, c_n \) such that

\[
0 = c\mathbf{y} + c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n \tag{2.6}
\]

We wish to show that each of these \( n+1 \) constants is zero. Suppose that \( c \) is not zero. Then we may solve (2.6) for \( \mathbf{y} \) as a linear combination of the vectors in \( A \), contradicting our assumption that \( \mathbf{y} \) is not in the span of \( A \). Thus \( c = 0 \). But then (2.6) reduces to a linear combination, of the vectors in \( A \), which equals the zero vector. Since \( A \) is linearly independent, each of the constants \( c_k = 0 \). Thus, the set \( \{\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_n\} \) is linearly independent.

The last lemma is also used to construct linearly independent spanning subsets of a vector space. We start out with a linearly independent subset. If its span is not \( V \), we add to it a vector \( \mathbf{y} \) not in its span. If the new set spans \( V \), we’re done. If not, another vector is added. This process continues until we reach a spanning set.

For most of the vector spaces we’ve seen either of the above procedures is successful in constructing bases. There are, however, some vector spaces for which this does not work. Those which are not finite dimensional. See Definition 2.10

Example 2. Let \( A = \{(1,1,-1),(0,1,1)\} \). Find a vector \( \mathbf{y} \) such that when \( \mathbf{y} \) is added to \( A \), we have a linearly independent spanning subset of \( \mathbb{R}^3 \).

\[
S[A] = \{c_1(1,1,-1) + c_2(0,1,1)\}
= \{(c_1, c_1 + c_2, c_2 - c_1)\}
= \{(x_1, x_2, x_3): x_2 - 2x_1 = x_3\}
\]

From this last description of \( S[A] \), it is easy to find vectors that are not in the span of \( A \). For example, \( (1,0,0) \) is one such vector. Since \( A \) is linearly independent, Lemma 2.2 now tells us that the set

\[
\{(1,0,0), (1,1,-1), (0,1,1)\}
\]

is also linearly independent. The reader may easily verify, and should do so, that this set does indeed span \( \mathbb{R}^3 \).

Definition 2.9. Let \( V \) be a vector space. A subset \( B \) of \( V \) is called a basis of \( V \) if it satisfies:

1. \( B \) is linearly independent.
2. \( B \) is a spanning set for \( V \); that is, every vector in \( V \) can be written as a linear combination of the vectors in \( B \).

It can be shown that every vector space containing more than the zero vector has a basis. We shall not prove this theorem for arbitrary spaces but instead will exhibit a basis for most of the vector spaces discussed in this text.

Example 3.
a. For $\mathbb{R}^1$, the vector space of real numbers, any set consisting of just one nonzero number is a basis.

b. In $\mathbb{R}^2$ let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$. The set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis of $\mathbb{R}^2$ and is the standard basis of $\mathbb{R}^2$.

c. If $V = \mathbb{R}^n$, define $\mathbf{e}_j, 1 \leq j \leq n$, by

\[
\begin{align*}
\mathbf{e}_1 &= (1, 0, \ldots, 0) \\
\mathbf{e}_2 &= (0, 1, 0, \ldots, 0) \\
& \vdots \\
\mathbf{e}_j &= (0, \ldots, 0, 1, 0, \ldots, 0), \text{ a 1 is in the } j\text{th slot.}
\end{align*}
\]

The set $\{\mathbf{e}_j : 1 \leq j \leq n\}$ is a basis for $\mathbb{R}^n$, as the reader can easily show. This particular set is the standard basis of $\mathbb{R}^n$. □

**Theorem 2.9.** Let $V$ be a vector space that has a basis consisting of $n$ vectors. Then:

a. Any set with more than $n$ vectors is linearly dependent.

b. Any spanning set has at least $n$ vectors.

c. Any linearly independent set has at most $n$ vectors.

d. Every basis of $V$ contains exactly $n$ vectors.

**Proof.** Part a. Suppose that $A = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is the given basis of $V$ with $n$ vectors. Let $B = \{\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}\}$ be any subset of $V$ with $n + 1$ vectors. We want to show that $B$ is linearly dependent. Since $A$ is a basis of $V$, it is a spanning set. Hence, there are constants $a_{jk}$ such that

\[
\mathbf{x}_k = a_{1k}\mathbf{f}_1 + \cdots + a_{nk}\mathbf{f}_n
\]

We want to find constants $c_k$, $1 \leq k \leq n + 1$, not all zero such that

\[
0 = c_1\mathbf{x}_1 + \cdots + c_{n+1}\mathbf{x}_{n+1} = \sum_{k=1}^{n+1} c_k \mathbf{x}_k
\]

\[
= \sum_{k=1}^{n+1} c_k \sum_{j=1}^{n} a_{jk} \mathbf{f}_j = \sum_{j=1}^{n} \left( \sum_{k=1}^{n+1} c_k a_{jk} \right) \mathbf{f}_j
\]

If we pick the constants $c_k$ in such a manner that each of the coefficients multiplying $\mathbf{f}_j$ is zero, this linear combination of the vectors in $B$ will equal the zero vector. In other words, we wish to find a nontrivial solution of the following system of equations:

\[
a_{j1}c_1 + a_{j2}c_2 + \cdots + a_{j(n+1)}c_{n+1} = 0
\]
for $j = 1, 2, \ldots, n$. This is a homogeneous system of equations with more unknowns $(n + 1)$ than equations $(n)$. Therefore, there is a nontrivial solution, and the set $B$ is linearly dependent. Notice that we did not use the fact that $A$ is also linearly independent. The only hypothesis needed to show the linear dependence of any set with $n + 1$ vectors was the existence of a spanning set with fewer than $n + 1$ vectors.

Part b. Let $B = \{x_1, \ldots, x_m\}$ be any spanning set of $V$. We wish to show that $B$ must contain at least $n$ elements. Suppose it doesn’t, i.e., $m$ is less than $n$. But then by part a, since $B$ is a spanning set, any set with more than $m$ vectors must be linearly dependent. In particular the set $A$ would have to be linearly dependent. This contradicts the assumption that $A$ is linearly independent.

Part c. Suppose $B = \{x_1, \ldots, x_p\}$ is linearly independent; then clearly part a implies that $p \leq n$. That is, any linearly independent set of vectors cannot have more than $n$ vectors in it.

Part d. Let $A_1 = \{g_1, \ldots, g_p\}$ be any other basis of $V$. By part b, since $A_1$ is spanning, we must have $p \geq n$. But reversing the roles of $A$ and $A_1$, we have by the same reasoning that $p \leq n$. Thus every basis of $V$ must have the same number of vectors. □

**Definition 2.10.** Let $V$ be a vector space. We say that the dimension of $V$ is $n$, $\dim(V) = n$, if $V$ has a basis of $n$ vectors. If $V$ is the vector space that consists of the zero vector only, we define the dimension of $V$ to be 0. If for any $n, V$ has a set of $n$ linearly independent vectors, we say that the dimension of $V$ is infinite.

Most of the vector spaces we discuss in this book will have a basis consisting of a finite number of vectors. There are, however, a large number of vector spaces that do not have such a basis. Two examples of such spaces are the set of all polynomials, and the set of all real-valued continuous functions defined on some interval of real numbers, i.e., $C[a, b]$. Both of these spaces are infinite-dimensional.

Example 3 exhibited a basis of $\mathbb{R}^n$ that contained $n$ vectors. Theorem 2.9 now tells us that any subset of $\mathbb{R}^2$ with more than two vectors must be linearly dependent. We also have that $\dim(\mathbb{R}^n) = n, n = 1, 2, \ldots$.

**Example 4.**

a. Show that $\dim(P_n) = n + 1$. Since $P_n$ consists of all polynomials with real coefficients of degree less than or equal to $n$, we have

$$P_n = \{a_n + a_1 t + \cdots + a_n t^n: \ a_k \text{ arbitrary real numbers}\}$$

We claim that the set $A = \{1, t, t^2, \ldots, t^n\}$ is a basis for $P_n$. It is clear that any polynomial of degree less than or equal to $n$ can be written as a linear combination of the vectors in $A$. We leave it to the reader to show that $A$ is also linearly independent. Since $A$ has $n + 1$ vectors, $\dim P_n = n + 1$. 
b. The vector space $M_{mn}$, consisting of all $m \times n$ matrices, has dimension equal to $mn$. The reader is asked in the problems to show that the matrices $E_{jk}$, $1 \leq j \leq m$, $1 \leq k \leq n$, which contain a 1 in the $j,k$ position and zeros everywhere else, form a basis for $M_{mn}$. □

**Example 5.** Consider the following system of homogeneous equations:

\[
\begin{align*}
2x_1 - x_3 + x_4 &= 0 \\
x_2 + x_4 &= 0
\end{align*}
\] (2.7)

Find a basis for the solution space.

**Solution.** The coefficient matrix

\[
\begin{bmatrix}
2 & 0 & -1 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]

is row equivalent to

\[
\begin{bmatrix}
1 & 0 & -\frac{1}{2} & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\].

Thus every solution to (2.7) must satisfy

\[
\begin{align*}
x_1 &= \frac{x_3 - x_4}{2} \\
x_2 &= -x_4
\end{align*}
\]

We have two free parameters $x_3$ and $x_4$ which we set equal to 1 and 0 to get one solution. To get a second solution, which is linearly independent of the first, we set $x_3$ and $x_4$ equal to 0 and 1, respectively. Thus we have the set $S = \{(\frac{1}{2},0,1,0), (-\frac{1}{2},-1,0,1)\}$ which is contained in the solution space of (2.7). Clearly $S$ is linearly independent. [The last two slots look like (1,0) and (0,1).] Moreover, every solution of (2.7) can be written as a linear combination of these two vectors; for suppose $(x_1,x_2,x_3,x_4)$ satisfies (2.7). Then we have

\[
(x_1,x_2,x_3,x_4) = \left(\frac{x_3 - x_4}{2}, -x_4, x_3, x_4\right)
\]

\[
= x_3 \left(\frac{1}{2},0,1,0\right) + x_4 \left(-\frac{1}{2},-1,0,1\right)
\]

Thus, $S$ is a basis for the solution space of (2.7), and the dimension of this subspace of $\mathbb{R}^4$ is 2. □

**Example 6.** Consider the system of linear equations

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - x_3 &= 0 \\
2x_1 + x_2 + x_3 &= 0
\end{align*}
\]

Find the dimensional of the solution space of this system.

**Solution.** The coefficient matrix

\[
\begin{bmatrix}
1 & -2 & 1 \\
0 & 1 & -1 \\
2 & 1 & 1
\end{bmatrix}
\]
of this system is row equivalent to

\[
\begin{bmatrix}
1 & -2 & 1 \\
0 & 1 & -1 \\
0 & 0 & 4
\end{bmatrix}
\]

This second system has only the trivial solution. Therefore, the solution space \( K \) consists of just the zero vector and we have \( \dim(K) = 0 \). □

**Example 7.** Let \( V = \mathbb{R}^2 \). Let \( W = \{(x_1, x_2, x_3) : x_3 = 2x_1 + x_2 \} \). Find a basis for \( W \) and then extend it to a basis of \( V \); i.e., add enough vectors to the basis of \( W \) to get a basis for \( V \).

**Solution.** Since \( W = \{(x_1, x_2, 2x_1 + x_2) : x_1 \text{ and } x_2 \text{ arbitrary}\} \), there are two free parameters \( x_1 \) and \( x_2 \). Let’s alternately set them equal to 0 and 1 to get a basis for \( W \). This gives us

\[
\begin{align*}
f_1 &= (1, 0, 2) : x_1 = 1, x_2 = 0 \\
f_2 &= (0, 1, 1) : x_1 = 0, x_2 = 1
\end{align*}
\]

Both \( f_1 \) and \( f_2 \) belong to \( W \), and they are linearly independent. To see that they span \( W \), let \( x \) be in \( W \). Then for some \( x_1 \) and \( x_2 \)

\[
x = (x_1, x_2, 2x_1 + x_2) = (x_1, 0, 2x_1) + (0, x_2, x_2)
\]

\[
= x_1(1, 0, 2) + x_2(0, 1, 1) = x_1f_1 + x_2f_2
\]

Since the set \( \{f_1, f_2\} \) is linearly independent and spans \( W \), it is a basis. We now face the problem of extending this basis to one for \( \mathbb{R}^3 \). Since \( \dim(\mathbb{R}^3) = 3 \), we need one additional vector. Thus, we want to find a vector \( y \) such that the set \( \{f_1, f_2, y\} \) is linearly independent. Let \( y = (1, 1, 0) \). One easily sees that \( y \) is not in \( W \). Hence, by Lemma 2.2, we may conclude that \( \{f_1, f_2, y\} \) is linearly independent. We leave it for the reader to show that this set spans \( \mathbb{R}^3 \). (Hint: Suppose not. Then we should be able to find a linearly independent subset of \( \mathbb{R}^3 \) with four vectors in it! See c. of Theorem 2.9.) □

Figure 2.13 shows \( f_1, f_2, \) and \( y \). It is clear that \( W = \text{span}\{f_1, f_2\} \) is the plane in \( \mathbb{R}^3 \) passing through the three points \((1,0,2), (0,1,1), \) and \((0,0,0)\). The vector \( y = (1,1,0) \) clearly does not lie in this plane.

The preceding example used Theorem 2.10 to infer that a certain set is a basis.

**Theorem 2.10.** Let \( V \) be a vector space with \( \dim(V) = n \). Then

a. Any linearly independent set with \( n \) vectors is a basis of \( V \).

b. Any spanning set with \( n \) vectors is a basis of \( V \).

**Proof.** Suppose first that \( A \) is a linearly independent set of \( n \) vectors such that \( \text{span}\{A\} \) is not all of \( V \). Then there must be a vector \( y \) not in \( \text{span}\{A\} \). But then
Lemma 2.2 implies that the set consisting of $A$ and $y$ must be linearly independent. This contradicts a. of Theorem 2.9. Therefore, any linearly independent set with $n$ vectors must also span the vector space. Hence it is a basis. Suppose next that $A$ is a spanning set of $n$ vectors. Then, by Lemma 2.1, if $A$ is not linearly independent, we may discard a vector from $A$, obtaining a set with fewer than $n$ vectors, and which still spans $V$. This contradicts b. of Theorem 2.9. \[\square\]

**Example 8.** Let $A = \begin{bmatrix} 3 & 2 & 16 & 23 \\ 2 & 1 & 10 & 13 \end{bmatrix}$. Find a basis for the solution space of $AX = 0$. If this is not a basis for $\mathbb{R}^4$, find a basis of $\mathbb{R}^4$ that contains the basis of the solution space.

**Solution.** The matrix $A$ is row equivalent to the matrix

\[
\begin{bmatrix}
1 & 0 & 4 & 3 \\
0 & 1 & 2 & 7
\end{bmatrix}
\]

Thus, if $X = (x_1, x_2, x_3, x_4)^T$ solves $AX = 0$, we must have $x_2 = -2x_3 - 7x_4$ and $x_1 = -4x_3 - 3x_4$. Setting $x_3 = 1$, $x_4 = 0$, and then $x_3 = 0$, $x_4 = 1$, we have two linearly independent solutions to our system:

\[
f_1 = (-4, -2, 1, 0) \quad \text{and} \quad f_2 = (-3, -7, 0, 1)
\]

Clearly the set $\{f_1, f_2\}$ is a basis for the solution space. To extend this set to a basis of $\mathbb{R}^4$ we have to find two more vectors $g_1$ and $g_2$ such that $\{f_1, f_2, g_1, g_2\}$ is linearly independent. $\mathbb{R}^4$ has as a basis $\{e_1, e_2, e_3, e_4\}$ where $e_1 = (1, 0, 0, 0)$, etc. Two of these $e_k$’s will not depend linearly on $f_1$ and $f_2$. We need to find such a pair. A quick inspection of $f_1$ and $f_2$ shows us that $e_1$ and $e_2$ will work. We leave the details of showing that $\{f_1, f_2, e_1, e_2\}$ is linearly independent to the reader. \[\square\]
We next outline a systematic procedure that can be used to extend any linearly independent set to a basis. Suppose \( A = \{x_1, \ldots, x_r\} \) is a linearly independent subset of a vector space \( V \), where \( \dim(V) = n \) and \( r < n \). We need to find \( n - r \) vectors to adjoin to \( A \) to form a basis. Suppose \( \{f_1, \ldots, f_n\} \) is any basis of \( V \). Consider the set \( \{x_1, \ldots, x_r, f_1, \ldots, f_n\} \). It is linearly dependent and its span is all of \( V \). Now just discard those \( f_k \) which depend on the \( x_j \) and the remaining \( f_p \), until \( r \) vectors have been discarded. The vectors that are left form a basis containing \( A \).

**Example 9.** Extend the linearly independent set of vectors \( B = \{(2,0,-1,0,0), (1,1,0,1,1)\} \) to a basis of \( \mathbb{R}^5 \).

**Solution.** Let \( A \) be the \( 7 \times 5 \) matrix whose first two rows are the vectors in \( B \) and whose last five rows are the standard basis vectors of \( \mathbb{R}^5 \).

\[
A = \begin{bmatrix}
2 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Now use elementary row operations but not row interchanges to find a matrix that is row equivalent to \( A \) and that has two rows of zeros. Zeroing out rows corresponds to discarding those vectors in the standard basis which depend linearly on the remaining vectors. After a few row operations, we have \( A \) row equivalent to the matrix

\[
\begin{bmatrix}
2 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus the set \( \{(2,0,-1,0,0), (1,1,0,1,1), e_1, e_2, e_4\} \) is a basis of \( \mathbb{R}^5 \) that contains our original set \( A \). \( \square \)

**Example 10.** Find a basis for \( V = \{p: p \text{ is in } P_2, p(1) = 0\} \). If \( p \) is in \( P_2 \), then \( p(t) = a_0 + a_1 t + a_2 t^2 \), and \( p(1) = 0 \) implies

\[ p(1) = 0 = a_0 + a_1 + a_2 \]

Thus, \( F = \{1 - t^2, t - t^2\} \) is a subset of \( V \). \( F \) is easily shown to be linearly independent. To see that \( F \) spans \( V \), let \( p \) be in \( V \). Then

\[
p(t) = a_0 + a_1 t + a_2 t^2
= a_0 + a_1 t + (-a_0 - a_1)t^2
= a_0(1 - t^2) + a_1(t - t^2)
\]
Thus, \( p(t) \) is in \( S[F] \); \( F \) is a basis of \( V \) and \( \dim(V) \) equals 2. \( \square \)

**Problem Set 2.5**

1. Find a basis for \( S[(2, 6, 0, 1), (1, 1, 0, -1, 4), (0, 0, 0, 1, 1)] \).

2. Let \( W = \{ (x_1, x_2, x_3, x_4) : x_1 - x_2 = x_3 \} \). Show that \( W \) is a subspace of \( \mathbb{R}^4 \) and find a basis of \( W \).

3. Let \( K \) be the solution space of the following homogeneous system of equations:
   \[
   \begin{align*}
   2x_1 - 6x_2 + 3x_3 &= 0 \\
   x_2 + x_3 + x_4 &= 0
   \end{align*}
   \]
   Find any two linearly independent vectors in \( K \). Show that every other vector in \( K \) can be written as a linear combination of your two vectors, and conclude that these two vectors are a basis of \( K \). Extend your basis to a basis of \( \mathbb{R}^3 \).

4. Show that any subset of \( \mathbb{R}^3 \) containing at most two vectors will not span all of \( \mathbb{R}^3 \).

5. Find a basis of \( \mathbb{R}^3 \) that is not the standard one given in the text.

6. Find a basis of \( P_2 \) that is different from the standard basis \( \{1, t, t^2\} \).

7. Let \( K = \{ (x_1, x_2, x_3) : 2x_1 + x_2 = 0, x_1 + x_2 - x_3 = 0 \} \).
   a. Show that every vector in \( K \) can be written as a scalar multiple of \((-1, 2, 1)\). Thus, \( \dim(K) = 1 \).
   b. Find two vectors \( y_1 \) and \( y_2 \) such that \( \{(-1, 2, 1), y_1, y_2\} \) is a basis of \( \mathbb{R}^3 \).

8. Show that the matrices \[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\] form a basis of \( M_{22} \), the set of \( 2 \times 2 \) matrices.

9. Show that the \( m \times n \) matrices \( E_{ij} \) described in Example 4b form a basis of \( M_{mn} \), the set of \( m \times n \) matrices. Since there are \( mn \) such matrices, the dimension of the set of \( m \times n \) matrices is \( mn \).

10. Let \( K = \{ (x_1, x_2, x_3) : x_1 - x_2 + 3x_3 = 0 \} \).
    a. Find a basis for \( K \).
    b. \( \dim(K) = ? \)
    c. Extend your basis of \( K \) to a basis of \( \mathbb{R}^3 \).

11. Consider the complex numbers as a real vector space. Remember complex numbers are things of the form \( a + bi \), where \( i^2 = -1 \) and \( a \) and \( b \) are arbitrary real numbers. Find a basis for the complex numbers.
12. Let \( K = \{(x_1, x_2, x_3) : x_1 - x_2 = x_3, x_2 + 2x_3 = 0\}\).
   a. Find a basis \( B \) for \( K \).
   b. \( \dim(K) = ? \)
   c. How many vectors have to be added to \( B \) to get a basis for \( \mathbb{R}^3 \)?

13. Let \( V \) be the vector space of all \( n \times n \) matrices. Then we know that \( \dim(V) = n^2 \). Find a basis and determine the dimension of each of the following subspaces of \( V \).
   a. Scalar matrices
   b. Diagonal matrices
   c. Upper triangular matrices
   d. Lower triangular matrices

14. Verify that the subset \( A = \{e_k : k = 1, \ldots, n\} \) is a basis of \( \mathbb{R}^n \); cf. Example 3.

15. Show that the set \( A = \{(1, 0, 0), (1, 1, -1), (0, 1, 1)\} \) spans \( \mathbb{R}^3 \); cf. Example 2. Is \( A \) a basis of \( \mathbb{R}^3 \)?

16. Show that the subset \( \{1, t, \ldots, t^n\} \) is a basis of \( P_n \).

17. In problem 19 of Section 2.4 we defined what it means to say that an infinite set of vectors is linearly independent. Using that definition, we say that \( B \) is a basis of a vector space \( V \), if \( B \) is linearly independent and if \( S[B] \) equals \( V \). Again, the span of a set is all possible finite linear combinations of vectors from that set. Let \( V = C[0, 1] \). Let \( B = \{1, e^t, e^{2t}, \ldots, e^{nt}, \ldots\} \). If \( B \) is a basis for \( V \)?

18. Let \( V \) be the set of all polynomials with real coefficients. Let \( B = \{1, t, \ldots, t^n, \ldots\} \). Is \( B \) a basis for \( V \)?

19. Let \( V \) be a three-dimensional space. Let \( \mathbf{x} \) be any nonzero vector in \( V \).
   a. Show that there are vectors \( \mathbf{y}_1 \) and \( \mathbf{y}_2 \) such that \( \{\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2\} \) is a basis of \( V \).
   b. Show that any linearly independent subset \( A \) of \( V \) can be extended to a basis of \( V \).

20. Let \( V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c = 0 \right\} \). Find a basis for \( V \) and thus determine \( \dim(V) \).

21. Let \( V = \{ \mathbf{p} : \mathbf{p} \) is in \( P_4 \), \( \int_0^1 \mathbf{p}(t)dt = 0, \mathbf{p}'(2) = 0 \} \). Find a basis for \( V \).

22. Show that \( \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\} \) is a basis for \( M_{22} \).

23. Is \( \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \) a basis of \( M_{22} \)?
24. Let $V = \{ p : p \text{ is in } P_4, \int_0^1 p(t) dt = 0 = \int_1^2 p(t) dt \}$. Find a basis for $V$.

25. Let $A = \{(1, 0, 0), (0, 1, 1), (0, 2, 3), (0, 3, 4)(1, -1, 3)\}$.
   a. Show that $A$ is linearly dependent.
   b. What is the largest number of vectors in $A$ that can form a linearly independent set?
   c. For which vectors $x$ in $A$ is it true that $S[A] = S[A \setminus \{x\}]$?
   d. Find a subset $B$ of $A$ such that $B$ is linearly independent and for which $S[B] = S[A]$.

2.6 Coordinates of a Vector

We have seen in an earlier section (Theorem 2.7) that given a basis for a vector space, every vector can be uniquely written as a linear combination of the basis vectors. The constants that are used in this sum are called the coordinates of the vector with respect to the basis. In this section we show how these coordinates change when we change our bases. For convenience, we again state the definition of the coordinates of a vector.

**Definition 2.11.** Let $B = \{f_1, f_2, \ldots, f_n\}$ be a basis of a vector space $V$, $\dim(V) = n$. Then the coordinates of a vector $x$ in $V$ with respect to the basis $B$ are the constants needed to write $x$ as a linear combination of the basis vectors. Thus if 

$$x = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

then the coordinates of $x$ with respect to $B$ are the scalars $c_1, c_2, \ldots, c_n$. We write this as 

$$[x]_B = [c_1, c_2, \ldots, c_n]$$

Note that the ordering of the vectors in $B$ is important. If we change this order, we change the order in which we list the coordinates.

**Example 1.** Let $V = \mathbb{R}^3$.

a. Let $S = \{e_1, e_2, e_3\}$. Find the coordinates of $x = (1, -1, 2)$ with respect to $S$. Clearly,

$$x = (1, -1, 2) = (1, 0, 0) - (0, 1, 0) + 2(0, 0, 1)$$

Thus,

$$[x]_S = [1, -1, 2]$$

Note that since $S$ is the standard basis, the coordinates of $x$ with respect to $S$ are the same three numbers used to define $x$.

b. Let $G = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$. Thus, $G$ is just a reordering of the standard basis of $\mathbb{R}^3$. Then 

$$[x]_G = [(1, -1, 2)]_G = [2, -1, 1]$$
c. Let $G = \{(2, -1, 3), (0, 1, 1), (1, -1, 0)\}$. Find the coordinates of $\mathbf{x} = (1, -1, 2)$ with respect to $G$. We need to find constants $c_1, c_2,$ and $c_3$ such that

$$(1, -1, 2) = c_1(2, -1, 3) + c_2(0, 1, 1) + c_3(1, -1, 0)$$

The system of equations derived from this vector equation is

$$
\begin{align*}
1 &= 2c_1 + c_3 \\
-1 &= -c_1 + c_2 - c_3 \\
2 &= 3c_1 + c_2
\end{align*}
$$

The solution is $c_1 = 1, c_2 = -1, c_3 = -1$. Hence

$$[\mathbf{x}]_G = [1, -1, -1]$$

We now discuss the general situation. Let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ and $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_n\}$ be any two bases of a vector space $V$. Then each vector in $F$ can be written uniquely in terms of the vectors in $G$ and each vector in $G$ can be written uniquely in terms of the basis $F$. Let $[p_{jk}], 1 \leq j, k \leq n$, be such that

$$\mathbf{f}_j = p_{1j}\mathbf{g}_1 + p_{2j}\mathbf{g}_2 + \cdots + p_{nj}\mathbf{g}_n \tag{2.8}$$

Thus $[\mathbf{f}_j]_G = [p_{1j}, p_{2j}, \ldots, p_{nj}]$ for $1 \leq j \leq n$. Let $[q_{jk}], 1 \leq j, k \leq n$, be such that

$$\mathbf{g}_j = q_{1j}\mathbf{f}_1 + q_{2j}\mathbf{f}_2 + \cdots + q_{nj}\mathbf{f}_n \tag{2.9}$$

That is, $[\mathbf{g}_j]_F = [q_{1j}, q_{2j}, \ldots, q_{nj}]$.

Thus, the first column of the $n \times n$ matrix $P = [p_{jk}]$ consists of the coordinates of $\mathbf{f}_1$ and the $j$th column of $P$ consists of the coordinates of $\mathbf{f}_j$, both with respect to the basis $G$. Similarly, the $k$th column of $Q = [q_{jk}]$ is formed using the coordinates of $\mathbf{g}_k$ with respect to the basis $F$.

Since $P$ gives the coordinates of the vectors in $F$ with respect to $G$ and $Q$ gives the coordinates of the vectors in $G$ with respect to $F$, these two matrices should be related to each other. In fact, we have

**Theorem 2.11.** Let $F$ and $G$ be two bases of a vector space $V$. Let the matrices $P$ and $Q$ be as defined above. Then

$$Q = P^{-1}$$

**Proof.** From (2.8) and (2.9) we have

$$\mathbf{f}_j = \sum_{k=1}^{n} p_{kj}\mathbf{g}_k = \sum_{k=1}^{n} p_{kj} \left( \sum_{i=1}^{n} q_{ik}\mathbf{f}_i \right)$$

$$= \sum_{i=1}^{n} \left( \sum_{k=1}^{n} q_{ik}p_{kj} \right) \mathbf{f}_i$$
Thus the terms \( \sum_{k=1}^{n} q_{ik} p_{kj} \) are the coordinates of \( f_j \) with respect to \( F \). But clearly

\[
f_j = 0f_1 + 0f_2 + \cdots + (1)f_j + \cdots + 0f_n \quad \text{for} \quad j = 1, 2, \ldots, n
\]

Thus

\[
\sum_{k=1}^{n} q_{ik} p_{kj} = \delta_{ij} \quad (2.10)
\]

But this summand is the \( i, j \) entry of the matrix \( QP \). Hence we have \( QP = I_n \). In a similar manner we can show that \( PQ = I_n \). Therefore, \( Q = P^{-1} \). \( \square \)

In the following we refer to \( P \) or \( P^{-1} \) as the change of basis matrix. Which one is meant will be clear from the context.

**Example 2.** Let \( F = \{(1,0,0),(0,1,0), (0,0,1)\} \). Let \( G = \{(2, -1, 3), (0,1,1), (1,-1,0)\} \). Thus \( f_1, f_2, \) and \( f_3 \) are our standard basis vectors, and \( g_1 = (2, -1, 3), g_2 = (0,1,1), \) and \( g_3 = (1,-1,0) \).

We clearly have

\[
[g_1]_F = [2, -1, 3] \quad [g_2]_F = [0, 1, 1] \quad [g_3]_F = [1, -1, 0]
\]

Thus, the matrix \( Q = P^{-1} \) must equal

\[
\begin{bmatrix}
2 & 0 & 1 \\
-1 & 1 & -1 \\
3 & 1 & 0
\end{bmatrix}
\]

We next want to find the coordinates of \( f_j \) with respect to the basis \( G \). There are two ways to go: one is to actually find the \( p_{ij} \) such that \( f_1 = p_{11}g_1 + p_{21}g_2 + p_{31}g_3 \), etc., or we can just compute \( Q^{-1} \). The first way is instructive; so we will find the first column of \( P \) by computing the coordinates of \( f_1 \) with respect to \( G \). We want to find constants \( c_k, k = 1, 2, 3 \), such that

\[
(1,0,0) = c_1(2, -1, 3) + c_2(0,1,1) + c_3(1,-1,0)
\]

The solutions are \( c_1 = -\frac{1}{2}, c_2 = \frac{3}{2}, c_3 = 2 \). Thus \( (1,0,0) = (-\frac{1}{2})g_1 + (\frac{3}{2})g_2 + 2g_3 \).

If we write out the above equations for the \( c_i \) as a matrix equation we have

\[
\begin{bmatrix}
2 & 0 & 1 \\
-1 & 1 & -1 \\
3 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

or

\[
Q \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]
Thus, \((c_1 \ c_2 \ c_3)^T = Q^{-1} (1 \ 0 \ 0)^T\), and clearly the first column of \(P\) must be the first column of \(Q^{-1}\), and

\[
Q^{-1} = \begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\
2 & 1 & -1
\end{bmatrix}
\]

Reading down the columns of \(Q^{-1} = P\) we have

\[
f_1 = \left(\begin{array}{c}
-\frac{1}{2}
\end{array}\right) g_1 + \left(\begin{array}{c}
\frac{3}{2}
\end{array}\right) g_2 + 2g_3
\]
\[
f_2 = \left(\begin{array}{c}
-\frac{1}{2}
\end{array}\right) g_1 + \left(\begin{array}{c}
\frac{3}{2}
\end{array}\right) g_2 + g_3
\]
\[
f_3 = \left(\begin{array}{c}
\frac{1}{2}
\end{array}\right) g_1 + \left(\begin{array}{c}
-\frac{1}{2}
\end{array}\right) g_2 + (-1)g_3
\]

We next discuss how to use the change of basis matrix \(P\) to express the coordinates of \(x\) with respect to \(G\) in terms of the coordinates of \(x\) with respect to \(F\). Thus let \(F = \{f_1, \ldots, f_n\}\) and \(G = \{g_1, \ldots, g_n\}\). Suppose \([x]_F = [x_1, \ldots, x_n]\). Then

\[
x = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n
\]

\[
x = x_1 \left( \sum_{j=1}^{n} p_{1j} g_j \right) + x_2 \left( \sum_{j=1}^{n} p_{2j} g_j \right) + \cdots + x_n \left( \sum_{j=1}^{n} p_{nj} g_j \right)
\]

Thus,

\[
[x]_G = \left[ \sum_{k=1}^{n} p_{1k} x_k, \sum_{k=1}^{n} p_{2k} x_k, \ldots, \sum_{k=1}^{n} p_{nk} x_k \right]
\]

(2.11)

Note that the \(j\)th coordinate of \(x\) with respect to \(G\) is the \(j\)th row of the matrix product \(P([x_1, \ldots, x_n])^T\). Thus, (2.11) implies the following equation:

\[
[x]_G^T = P[x]_F^T
\]

(2.12)

In the sequel we use equation (2.12) to specify \(P\); that is, we ask for the change of basis matrix \(P\) such that \([x]_G^T = P[x]_F^T\). The following summarizes these
relationships:

\[ F = \{ f_1, \ldots, f_n \} \quad G = \{ g_1, \ldots, g_n \} \]

\[ f_j = \sum_{k=1}^{n} p_{kj} g_k \quad g_j = \sum_{k=1}^{n} q_{kj} f_k \]

\[ [x]_F = [x_1, \ldots, x_n] \quad [x]_G = [y_1, \ldots, y_n] \]

\[ y_j = \sum_{k=1}^{n} p_{jk} x_k \]

\[ [x]_G^T = P [x]_F^T \quad [x]_F^T = P^{-1} [x]_G^T \]

One way of remembering how to compute a change of basis matrix is to realize that we really want a matrix \( P \) such that, given two bases \( F \) and \( G \), the coordinates of any vector \( x \) with respect to \( F \) and \( G \) are related by equation (2.12):

\[ [x]_G^T = P [x]_F^T \quad (2.12) \]

Let us start from this equation and see what \( P \) must equal. Let \( F = \{ f_1, \ldots, f_n \} \). If (2.12) holds for all \( x \) in \( V \), it must also hold for \( x = f_1 \). What are the coordinates of \( f_1 \) with respect to \( F \)? Clearly we have

\[ [f_1]_F = [1, 0, \ldots, 0] \]

But then \( P [f_1]_F^T \) is just the first column of \( P \); that is, the first column of \( P \) equals \( [f_1]_G^T \). This is, as we would expect, equation (2.8) for \( j = 1 \). Similarly we see that the \( k \)th column of \( P \) must equal \( [f_k]_G^T \).

**Example 3.** Let \( V \) be a vector space. Let \( F_1, F_2, \) and \( F_3 \) each denote a basis of \( V \). Suppose \( P \) and \( Q \) denote change of basis matrices between \( F_1 \) and \( F_2 \), and between \( F_3 \) and \( F_2 \), respectively, such that

\[ [x]_{F_1}^T = P [x]_{F_2}^T \quad (2.13) \]

and

\[ [x]_{F_2}^T = Q [x]_{F_3}^T \quad (2.14) \]

Find the change of basis matrix \( R \), relating \( F_1 \) and \( F_3 \), such that

\[ [x]_{F_3}^T = R [x]_{F_1}^T \]

From (2.13) and (2.14) we have \( [x]_{F_3}^T = Q [x]_{F_2}^T = Q P^{-1} [x]_{F_1}^T \). By problem 11 at the end of this section we must have

\[ R = Q P^{-1} \]

The reader might find this fact useful when computing change of basis matrices, when neither basis is the standard basis. We illustrate this below.
Example 4. Let $V = \mathbb{R}^2$. Let $F = \{(7, 8), (-9, 20)\}$. Let $G = \{(6, -5), (1, 1)\}$. Find the change of basis matrix $R$ such that

$$[x]_G^F = R[x]_F^G.$$ 

Let $S$ be the standard basis of $\mathbb{R}^2$. Let $P$ be such that $[x]_S^F = P[x]_F^G$. Let $Q$ be such that $[x]_S^G = Q[x]_G^F$. Thus, the columns of $P$ and $Q$ are the coordinates of the vectors in $F$ and $G$, respectively, with respect to the standard basis. Hence,

$$P = \begin{bmatrix} 7 & -9 \\ 8 & 20 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix}$$

and we have

$$[x]_G^F = Q^{-1}[x]_S^F = Q^{-1}(P[x]_F^G) = (Q^{-1}P)[x]_F^G.$$ 

Thus,

$$R = Q^{-1}P = \begin{bmatrix} 7 & -9 \\ 8 & 20 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix}$$

To check our computations, we verify that the first column of $R$ consists of the coordinates of $f_1$ with respect to $G$

$$-\left(\frac{1}{11}(6, -5)\right) + \left(\frac{83}{11}\right)(1, 1) = \begin{bmatrix} 77/11 \\ 88/11 \end{bmatrix} = (7, 8) = f_1. \quad \square$$

Problem Set 2.6

1. Find the coordinates of $x = (1, -1, 2)$ with respect to each of the following bases:

   a. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
   b. $\{(1, 1, 1), (2, 1, 4), (-1, 1, 1)\}$
   c. $\{(2, 0, 6), (4, 2, 0), (0, 3, 2)\}$

2. Find the coordinates of $(1, 2, 0, -4)$ with respect to each of the following bases:

   a. The standard basis of $\mathbb{R}^4$
   b. $\{(0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0)\}$
3. Find the coordinates of \( x = (6, -4) \) with respect to each of the following bases of \( \mathbb{R}^2 \):
   a. \( \{ (1,0), (0,1) \} \)
   b. \( \{ (6,-4), (9,17) \} \)
   c. \( \{ (1,1), (-1,0) \} \)
   d. \( \{ (9,17), (6,-4) \} \)

4. Let \( F = \{ (1,-2), (8,3) \} \)
   a. If \( [x]_F = [1,0] \), then \( x = ? \)
   b. If \( [x]_F = [1,1] \), then \( x = ? \)
   c. If \( [x]_F = [0,0] \), then \( x = ? \)
   d. If \( x = (1,1) \), then \( [x]_F = ? \)

5. Let \( V = M_{22} \). Let \( F \) be the set:
   \[
   \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}
   \]
   a. Show that \( F \) is a basis of \( V \).
   b. Let \( x = \begin{bmatrix} 6 \\ 3 \\ -4 \\ 2 \end{bmatrix} \); \( [x]_F = ? \)
   c. If \( [x]_F = [0,1,0,0] \), then \( x \) equals?

6. Let \( V = P_2 \). Let \( F = \{ 1, 1 + t, 1 + t + t^2 \} \).
   a. Show that \( F \) is a basis of \( V \).
   b. If \( [x]_F = [-2,3,7] \), then \( x \) equals?
   c. \( [6 - t^2]_F = ? \)

7. Let \( S \) be the standard basis of \( \mathbb{R}^2 \), and let \( G = \{ (1,6), (2,3) \} \).
   a. Show that \( G \) is a basis.
   b. Find the change of basis matrix \( P \) such that \( [x]_G^T = P[x]_S^T \).
   c. Find the change of basis matrix \( Q \) such that \( [x]_S^T = Q[x]_G^T \).
   d. Compute \( PQ \) and \( QP \).

8. Let \( x = (-8,4) \). Let \( S \) and \( G \) be the bases in problem 7.
   a. Find \( [x]_S \) and \( [x]_G \).
   b. Show that \( [x]_G^T = P[x]_S^T \).

9. Let \( F = \{ (-1,7), (2,-3) \} \) and \( G = \{ (1,2), (1,3) \} \).
   a. Show that both \( F \) and \( G \) are bases of \( \mathbb{R}^2 \).
   b. Find the change of basis matrix \( P \) such that \( [x]_G^T = P[x]_F^T \).

10. Let \( x = (3,7) \). Let \( F \) and \( G \) be the bases in problem 9.
2.6. COORDINATES OF A VECTOR

a. Find \( [x]_F \) and \( [x]_G \).

b. Show that \( [x]_G^T = P[x]_F^T \).

11. Let \( A \) and \( B \) be the two \( m \times n \) matrices. Suppose that \( Ax = Bx \) for every \( x \) in \( \mathbb{R}^n \). Show \( A = B \). Note that it will be sufficient to show that if \( Ax = 0 \) for every \( x \) in \( \mathbb{R}^n \), then \( A \) must be the \( m \times n \) zero matrix. Why?

12. Let \( F = \{ (1, 1), (-1, 2) \} \), \( G = \{ (-1, 2), (1, 1) \} \).
   a. Find the change of basis matrix \( P \) such that \( [x]_G = P[x]_F \).
   b. If \( x = (1, 1) \), find \( [x]_F \) and \( [x]_G \).
   c. Show \( [x]_G = P^{-1}[x]_F \).

13. Let \( S \) be the standard basis of \( \mathbb{R}^3 \) and let \( G = \{ (6, 0, 1), (-1, -1, 0), (0, 3, 1) \} \).
   a. Show that \( G \) is a basis of \( \mathbb{R}^3 \).
   b. Find the change of basis matrix \( P \) such that \( [x]_G = P[x]_S \).
   c. Write \( e_k, k = 1, 2, 3 \), as linear combinations of the vectors in \( G \).

14. Let \( x = (5, -3, 4) \). Let \( S \) and \( G \) be the bases in problem 13. Find \( [x]_G \) in two different ways.

15. Let \( F = \{ f_1, f_2 \} \) be a basis of \( \mathbb{R}^2 \). Let \( G = \{ f_2, f_1 \} \). Find the change of basis matrix \( P \), such that \( [x]_G = P[x]_F \).

16. Let \( F \) be any basis of some \( n \)-dimensional vector space. Let \( G \) consist of the same vectors that belong to \( F \), but in perhaps a different order. Show that the matrix \( P \) relating these two bases is a permutation matrix; cf. problem 13 in Section 1.5.

17. Let \( V = M_{22} \). Let \( F = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \). Let \( G = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\} \).

   a. Let \( x = \begin{bmatrix} 2 \\ 7 \\ -6 \\ 5 \end{bmatrix} \). \( [x]_F = ? \), \( [x]_G = ? \).
   b. Find the change of basis matrix \( P \) such that \( [x]_G = P[x]_F \).
   c. Check that \( [x]_G = P[x]_F \) for the vector \( x \) of part a.
   d. If \( [x]_G = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \), then \( x \) equals?

18. Let \( V = P_3 \). Let \( F = \{ -2, t + t^2, t^2 - 1, t^3 + 1 \} \). Let \( S \) denote the standard basis of \( V \).
   a. Verify that \( F \) is a basis of \( V \).
   b. Find the change of basis matrix \( P \) such that \( [x]_S = P[x]_F \).
d. Compute the coordinates of \( t^3, t^2, t, \) and 1 with respect to the basis \( F \).

19. Let \( S = \{(1,0), (0,1)\} \) and let \( B = \{(2, -3), (1, 6)\} \).

a. Show that both \( S \) and \( B \) are bases of \( \mathbb{R}^2 \).

b. Find the coordinates of \((-6,3)\) with respect to \( S \).

c. Find the coordinates of \((-6,3)\) with respect to \( B \).

20. Let \( S \) and \( B \) be the same sets as in problem 19. The coordinates of \((2, -3)\) and \((1,6)\) with respect to \( S \) are \([2, -3]\) and \([1,6]\), respectively. Show that the coordinates of \((1,0)\) and \((0,1)\) with respect to \( B \) are \(\left[\frac{2}{5}, -\frac{3}{5}\right]\) and \(\left[-\frac{3}{5}, \frac{2}{5}\right]\), respectively. Let

\[
W = \begin{bmatrix}
\frac{2}{5} & -\frac{3}{5} \\
\frac{1}{5} & \frac{2}{5}
\end{bmatrix}
\text{ and } Q = \begin{bmatrix}
2 & 1 \\
-3 & 6
\end{bmatrix}
\]

a. Identify the columns of \( P \) and \( Q \) with the above coordinates.

b. Show that \( PQ = QP = I_2 \).

21. Let \( S \) and \( B \) be the sets defined in problem 19. In that problem, we saw that the coordinates of \((-6,3)\) with respect to \( B \) are \(\left[-\frac{13}{5}, -\frac{4}{5}\right]\). Show that

\[
\begin{bmatrix}
-\frac{13}{5} \\
-\frac{4}{5}
\end{bmatrix} = P \begin{bmatrix}
-6 \\
3
\end{bmatrix}
\]

22. Find the coordinates of \((2,6)\) with respect to each of the following bases of \( \mathbb{R}^2 \):

a. \(\{(1,0), (0,1)\}\) \hspace{1cm} b. \(\{(0,1), (1,0)\}\)

c. \(\{(2,6), (3,0)\}\) \hspace{1cm} d. \(\{(3,0), (2,6)\}\)

**Supplementary Problems**

1. Define each of the following terms and then give at least two examples of each one:

a. A vector space of dimension 5

b. Linearly independent set

c. Basis

d. A vector space of dimension 100 = 10^2

e. Spanning set

2. Let \( V \) be the vector space of all polynomials in \( t \). Show that there is no finite set of vectors in \( V \) that spans \( V \).
3. Let $F = \{(1, -1, 2), (0, 1, 1), (1, 1, 1)\}$.
   a. Show that $F$ is a basis of $\mathbb{R}^3$.
   b. Let $x = (1, 0, 0)$. Find $[x]_F$.
   c. Find the change of basis matrix $P$, relating $F$ to the standard basis, for which $[x]_F^T = P[x]_S^T$.
   d. Explain why the first column of $P$ is your answer to part b.

4. Show that any nontrivial subspace $W$ of a finite-dimensional vector space $V$ has a basis and $\dim(W) \leq \dim(V)$.

5. Let $V$ be the vector space consisting of all real-valued functions defined on $[0, 1]$. Which of the following subsets are subspaces, and which of these are finite-dimensional?
   a. All polynomials of degree $\leq 4$
   b. All $f(t)$ for which $f(\frac{1}{2}) = 0$
   c. All $f(t)$ such that $f(0) = f(1)$
   d. All $f(t)$ such that $f(0) + f(1) = 1$
   e. All functions of the form $ce^{rt}$ for some constants $c$ and $r$

6. Let $A$ be an $m \times m$ matrix. Show that $A$ is invertible if and only if its rows form a linearly independent set of vectors in $\mathbb{R}^m$. Hint: $A$ is invertible if and only if $A$ is row equivalent to $I_m$.

7. Suppose $X_1, \ldots, X_m$ are linearly independent vectors. Let $A = [a_{jk}]$ be an $m \times m$ matrix. Define

$$Y_j = \sum_{k=1}^{m} a_{kj}X_k \quad j = 1, \ldots, m$$

Show that the set of vectors $\{Y_1, \ldots, Y_m\}$ is linearly independent if and only if the matrix $A$ is invertible.

8. If $a, b,$ and $c$ are any three distinct constants, show that $\sin(a+t)$, $\sin(b+t)$, and $\sin(c+t)$ are linearly dependent on $[0,1]$.

9. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}$. Is $V$ a vector space?

10. Let $W$ be that subspace of $M_{22}$ spanned by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

a. Show that $\dim(W) = 2$ and that $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = d \text{ and } c = 0 \right\}$. 

b. Show that there is no real matrix $A$ such that

$$A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

11. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two vectors. Let $\mathbf{y}_1, \mathbf{y}_2,$ and $\mathbf{y}_3$ be any three vectors in $S[\mathbf{x}_1, \mathbf{x}_2]$. Show that these three vectors form a linearly dependent set.

12. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ be any set of vectors. Let $F$ be any set of $m$ vectors contained in the span of the given set of $r$ vectors. Show that if $m > r$, then $F$ must be a linearly dependent set of vectors.

13. Let $V = M_{22}$. Let $C = \{A: AB = BA \text{ for every } B \text{ in } V\}$. That is, $C$ consists of all $2 \times 2$ matrices that commute with every other $2 \times 2$ matrix.

   a. Show that $C$ is a subspace of $V$.
   
   b. Find a basis of $C$ and hence determine $\dim(C)$.
   
   c. Let $V = M_{nn}$, define $C$ as above, and repeat parts a and b.

14. For which numbers $x$ are the vectors $(2,3)$ and $(1,x)$ linearly independent?