Exercises Chapter VI.

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1. Calculate the lengths of the following vectors:

a. $||(1,2)|| = \sqrt{5}$. b. $||(-1,3,6)|| = \sqrt{1+9+36} = \sqrt{46}$

c. $||(1,1,2,8)|| = \sqrt{1+1+4+84} = \sqrt{90}$.

2. Find all unit vectors that are parallel to the vector $(1,2,-4)$.

If $\boldsymbol{x} = (x_1, x_2, x_3)$ is parallel to $(1,2,-4)$, then $(x_1, x_2, x_3) = \lambda(1,2,-4)$. The length of this vector is $|\lambda|\sqrt{21}$. If $\boldsymbol{x}$ is to have unit length then $\lambda = \pm 1/\sqrt{21}$. Thus,

$$\boldsymbol{x} = \pm \frac{(1,2,-4)}{\sqrt{21}}\,.$$

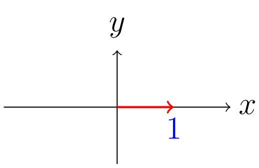
3. Compute the dot product of each of the following pairs of vectors:

a. $<(1,0),(0,1)>= 0$. b. $<(a,b),(b,a)>= 2ab$

c. $<(1,2,1),(3,-6,2)>= 3-12+2 = -7$

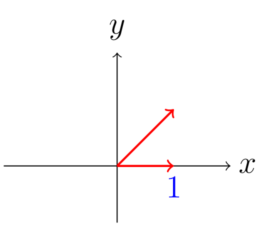
4. Sketch each of the following pairs of vectors. Compute their inner product and determine the cosine of the angle between them.

a. (1,0), (1,0)



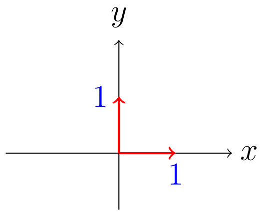
$<(1,0),(1,0)>= 1$, and $\cos\theta = \frac{1}{1\cdot 1} = 1\,.$

b. (1,0), (1,1)



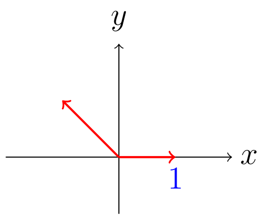
$<(1,0),(1,1)>= 1$ and $\cos\theta = \frac{1}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}\,.$

c. (1,0), (0,1)



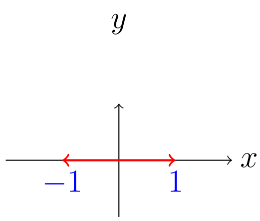
$< (1,0),(0,1) >= 0$ and $\cos\theta = 0$.

d. (1,0), $(-1,1)$



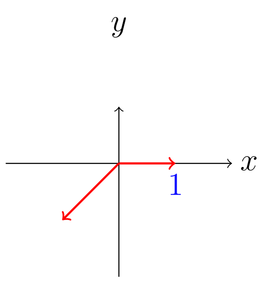
$< (1,0),(-1,1) >= -1$ and $\cos\theta = \frac{-1}{\sqrt{2}}$.

e. (1,0), $(-1,0)$



$< (1,0),(-1,0) >= -1$ and $\cos\theta = -1$.

f. (1,0), $(-1,-1)$



$< (1,0),(-1,-1) >= -1$ and $\cos\theta = \frac{-1}{\sqrt{2}}$.

5. Find the cosine of the angle between each of the following pairs of vectors:

a. (1,2), (3,-1), $\cos\theta = \dfrac{1}{\sqrt{5}\sqrt{10}} = \dfrac{1}{5\sqrt{2}}.$

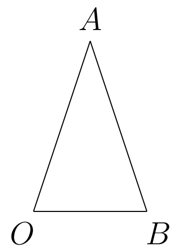
b. $(1, 0, -4)$, (6,1,2), $\cos\theta = \dfrac{-2}{\sqrt{17}\sqrt{41}}.$

c. $(-2, 3, 0, 1), (1, 2, 8, -2)$, $\cos\theta = \dfrac{-2}{\sqrt{14}\sqrt{73}}.$

16. Suppose that $\boldsymbol{x}$ is perpendicular to every vector in some set A. Show that $\boldsymbol{x}$ must then be perpendicular to every vector in $S[A]$.

Suppose a vector $\boldsymbol{y}$ is in $S[A]$. Then there are vectors $\{\boldsymbol{a}_i\}_{i=1}^k$ in A, and constants c_i such that $\boldsymbol{y} = \sum_{i=1}^k c_i \boldsymbol{a}_i$. Using the linearity of the inner product we have

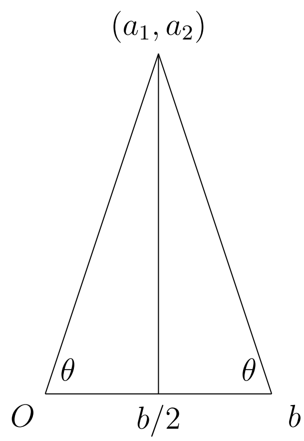
$$< \boldsymbol{x}, \boldsymbol{y} >=< \boldsymbol{x}, \sum_{i=1}^{k} c_i \boldsymbol{a}_i >= \sum_{i=1}^{k} c_i < \boldsymbol{x}, \boldsymbol{a}_i >= 0\,.$$

19. Let 

be an isosceles triangle with equal angles at O and B.

Show that the line drawn from the vertex A to the midpoint of OB is perpendicular to OB.

Assume that the point O is at the origin $(0,0)$. Draw the line from point A to the midpoint of line $\overline{OB}$. Let θ denote the value of the equal angles, and suppose A has coordinates (a_1, a_2), the midpoint has coordinates $(b/2, 0)$, and B has coordinates $(b, 0)$, with $a_1 > 0$, $a_2 > 0$, and $b > 0$. See the figure below.



We have the following expressions for the cosine of the angle θ:

$$\begin{aligned}
\cos\theta = \frac{< (a_1, a_2), (b/2, 0) >}{\sqrt{a_1^2 + a_2^2}(b/2)} &= \frac{< (a_1 - b, a_2), (-b/2, 0) >}{\sqrt{(a_1 - b)^2 + a_2^2}(b/2)} \\
\frac{a_1}{\sqrt{a_1^2 + a_2^2}} &= \frac{(b - a_1)}{\sqrt{(a_1 - b)^2 + a_2^2}} \\
0 &= a_2^2 b(b - 2a_1)
\end{aligned}$$

Since $a_2^2 b \neq 0$, we have $a_1 = b/2$. Thus, the vertex A sits directly about the midpoint B. From this it is clear that the line from A to the midpoint is perpendicular to the base of the triangle.

1. Compute $\text{Proj}_u\boldsymbol{x}$, where $\boldsymbol{x} = (7, -8)$ for each of the following unit vectors:

a. $(1, -2)/5^{1/2}$: $\quad$ $\text{Proj}_u\boldsymbol{x} = \frac{23}{5}(1, -2)$.

b. $(2, 3)/(13)^{1/2}$: $\quad$ $\text{Proj}_u\boldsymbol{x} = \frac{-10}{13}(2, 3)$.

c. $(1, 0)$: $\quad$ $\text{Proj}_u\boldsymbol{x} = 7(1, 0)$.

3. Let $\boldsymbol{x} = (7, -5)$. Let $U = \{(1, 5)/\sqrt{26},\ (-5, 1)/\sqrt{26}\}$.

a. Show that U is an orthonormal basis of $\mathbb{R}^2$.
Each vector in U has length 1, and the dot product of $(1, 5)$ with $(-5, 1)$ is zero. Thus, U is an orthonormal basis of $\mathbb{R}^2$.

b. Find $\text{Proj}_{u_j}\boldsymbol{x}$, where $\boldsymbol{u}_j$ is the jth unit vector in U.
The projections of $\boldsymbol{x}$ are:

$$\begin{aligned} \text{Proj}_{\boldsymbol{u}_1}\boldsymbol{x} &= \frac{-18}{\sqrt{26}}\boldsymbol{u}_1 \\ \text{Proj}_{\boldsymbol{u}_2}\boldsymbol{x} &= \frac{-40}{\sqrt{26}}\boldsymbol{u}_2 \end{aligned}$$

c. Compute the coordinates of $\boldsymbol{x}$ with respect to U.
The coordinates of $\boldsymbol{x}$ with respect to the basis U are:

$$[\boldsymbol{x}]_U = \left[\frac{-18}{\sqrt{26}}, \frac{-40}{\sqrt{26}}\right].$$

6. Let $V = \{(2, -3, 1), (2, 3, 5), (-9, -4, 6)\}$.

a. Show that V is an orthogonal set of vectors.
Just compute the various dot products. For future reference the lengths of the vectors in V are computed.

$$||\boldsymbol{v}_1|| = \sqrt{14}, \qquad ||\boldsymbol{v}_2|| = \sqrt{38}, \qquad ||\boldsymbol{v}_3|| = \sqrt{133}\,.$$

b. Let $\boldsymbol{x} = (7, -3, 4)$. Compute the projection of $\boldsymbol{x}$ onto the direction given by $\boldsymbol{v}_j$, where $\boldsymbol{v}_j$ is the jth vector in the set V.

$$\text{Proj}_{\boldsymbol{v}_1}\boldsymbol{x} = \frac{27}{14}\boldsymbol{v}_1, \qquad \text{Proj}_{\boldsymbol{v}_2}\boldsymbol{x} = \frac{25}{38}\boldsymbol{v}_2, \qquad , \text{Proj}_{\boldsymbol{v}_3}\boldsymbol{x} = \frac{-27}{133}\boldsymbol{v}_3,$$

c. Compute the coordinates of $(7, -3, 4) = \boldsymbol{x}$ with respect to the basis V.

$$[\boldsymbol{x}]_V = \left[\frac{27}{14}, \frac{25}{38}, \frac{-27}{133}\right].$$

7. Find the angle between the following pairs of vectors:

a. (1,1), (0,1) b. (1,1,1), (0,1,0)
c. (1,1,1,1), (0,1,0,0) d. $(6,7,-2,3),(-1,-2,1,1)$

a. $\cos\theta = \frac{1}{\sqrt{2}} \implies \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

b. $\cos\theta = \frac{1}{\sqrt{3}} \implies \theta = \arccos\frac{1}{\sqrt{3}} \approx 0.955\,\text{rad}$.

c. $\cos\theta = \frac{1}{\sqrt{4}} \implies \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$.

d. $\cos\theta = \frac{-19}{\sqrt{98}\sqrt{7}} \implies \theta = \arccos\frac{-19}{\sqrt{686}} \approx 2.382\,\text{rad}$.

10. Let $\boldsymbol{u}$ be an arbitrary unit vector in $\mathbb{R}^n$.

a. If $\boldsymbol{x}$ is the zero vector, show that $\text{Proj}_u\boldsymbol{x} = \boldsymbol{0}$.

$$\text{Proj}_u\boldsymbol{x} =< \boldsymbol{0}, \boldsymbol{u} > \boldsymbol{u} = 0\boldsymbol{u} = \boldsymbol{0}\,.$$

b. If $\boldsymbol{x}$ and $\boldsymbol{u}$ are perpendicular, show that $\text{Proj}_u\boldsymbol{x} = \boldsymbol{0}$.

$$\text{Proj}_u\boldsymbol{x} =< \boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u} = 0\boldsymbol{u} = \boldsymbol{0}\,.$$

c. Show that $\text{Proj}_u\boldsymbol{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$\begin{aligned}\text{Proj}_u(\boldsymbol{x}+\boldsymbol{y}) &= < \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{u} > \boldsymbol{u} =< \boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u}+ < \boldsymbol{y}, \boldsymbol{u} > \boldsymbol{u} = \text{Proj}_u\boldsymbol{x} + \text{Proj}_u\boldsymbol{y} \\ \text{Proj}_u(\alpha\boldsymbol{x}) &= < \alpha\boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u} = \alpha < \boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u} = \alpha\text{Proj}_u\boldsymbol{x}\end{aligned}$$

d. What is the dimension of the kernel of this linear transformation?
Since the dimension of the range of the projection mapping is one, the kernel must have dimension $n-1$.

11. Let $V = P_3$. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be any two polynomials in V. Define $\langle \boldsymbol{f}, \boldsymbol{g}\rangle = \int_0^1 \boldsymbol{f}(t)\boldsymbol{g}(t)dt$; cf. problem 8 in Section 6.1.

a. Find a unit vector $\boldsymbol{u}$ that points in the same direction as $\boldsymbol{f}(t) = t$.

$$\boldsymbol{u} = \frac{1}{||\boldsymbol{f}||}\boldsymbol{f} = \frac{1}{\left(\int_0^1 t^2\,dt\right)^{1/2}}t = \sqrt{3}\,t\,.$$

b. Find the projection of $\boldsymbol{t}^2$ onto the vector $\boldsymbol{u}$ of part a.

$$\text{Proj}_{\boldsymbol{u}}\boldsymbol{t}^2 = < t^2, \boldsymbol{u} > \boldsymbol{u} = \left(\int_0^1 t^2(\sqrt{3}\,t)\,dt\right)\sqrt{3}t = \frac{3}{4}t\,.$$

c. Find the cosine of the angle between the vectors $\boldsymbol{t}^2$ and $\boldsymbol{t}$.

$$\cos\theta = \frac{< t, t^2 >}{||t||\,||t^2||} = \frac{\int_0^1 t^3\,dt}{\left(\int_0^1 t^2\,dt\right)^{1/2}\left(\int_0^1 t^4\,dt\right)^{1/2}} = \frac{1/4}{\sqrt{1/3}\sqrt{1/5}} = \frac{\sqrt{15}}{4}$$

13. Let $V = P_2$. Define the inner product as we did in problems 11 and 12. Let $\boldsymbol{f}'$ denote the derivative of $\boldsymbol{f}$.

a. Find all polynomials in P_2 that are perpendicular to their derivatives.
The inner product of f with its derivative is

$$\begin{aligned} < \boldsymbol{f}, \boldsymbol{f}' > &= \int_0^1 f(t)\, f'(t)\, dt = \frac{1}{2}\int_0^1 (\frac{d}{dt} f^2)\, dt = \frac{1}{2}\left(f^2(1) - f^2(0)\right) \\ &= \frac{1}{2}\left(f(1) - f(0)\right)\left(f(1) + f(0)\right) \end{aligned}$$

Thus, $\boldsymbol{f}$ is orthogonal to its derivative if

$$f(1) - f(0) = 0 \quad \text{or} \quad f(1) + f(0) = 0$$

Note that this is a non-linear condition. So the subset of such polynomials will not necessarily be a subspace. The general polynomial in P_2 has the form $a_0 + a_1 t + a_2 t^2$. The two equations above say that $\boldsymbol{p}$ must have one of the following two forms:

$$p(t) = a_0 + a_1 t - a_1 t^2, \quad \text{or} \quad p(t) = a_o + a_1 t - (2a_0 + a_1)t^2$$

b. For any two polynomials f and g in V, compute $\langle \boldsymbol{f}, \boldsymbol{g}' \rangle + \langle \boldsymbol{f}', \boldsymbol{g} \rangle$.

$$\begin{aligned} \langle \boldsymbol{f}, \boldsymbol{g}' \rangle + \langle \boldsymbol{f}', \boldsymbol{g} \rangle &= \int_0^1 f(t) g'(t)\, dt + \int_0^1 f'(t) g(t)\, dt = \int_0^1 \frac{d}{dt}\left(f(t)g(t)\right)'\, dt \\ &= f(t)g(t)|_0^1 = f(1)g(1) - f(0)g(0) \end{aligned}$$

1. Use the Gram–Schmidt procedure to construct an orthonormal basis for each of the following subspaces of $\mathbb{R}^3$:

a. $W = \{(x_1, x_2, x_3)\colon\ x_1 - x_2 = 0\}$
One basis for W is $\{(1,1,0),(0,0,1)\}$. Using Gram-Schmidt we have

$$\begin{aligned}\boldsymbol{u}_1 &= \frac{1}{\sqrt{2}}(1,1,0) \quad \boldsymbol{v}_1 = (0,0,1) - \mathrm{Proj}_{\boldsymbol{u}_1}(0,0,1) = (0,0,1)\\ \boldsymbol{u}_2 &= \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||} = (0,0,1)\end{aligned}$$

Thus, an orthonormal basis of W is $\left\{\frac{1}{\sqrt{2}}(1,1,0),(0,0,1)\right\}$.

b. $W = S[(1,-1,2),(6,1,1)]$
Since the vectors used to generate (span) W are linearly independent they form a basis of W.

$$\begin{aligned}\boldsymbol{u}_1 &= \frac{1}{\sqrt{6}}(1,-1,2)\\ \boldsymbol{v}_1 &= (6,1,1) - \mathrm{Proj}_{\boldsymbol{u}_1}(6,1,1) = (6,1,1) - \frac{7}{6}(1,-1,2) = \frac{1}{6}(29,13,-8)\\ \boldsymbol{u}_2 &= \frac{1}{\sqrt{234}}(29,13,-8)\end{aligned}$$

An orthonormal basis of W is $\left\{\frac{1}{\sqrt{6}}(1,-1,2), \frac{1}{\sqrt{1074}}(29,13,-8))\right\}$.

2. Construct an orthonormal basis for $\mathbb{R}^3$ from the following basis,
$\{(0,5,1),\ (0,1,-5),\ (1,-2,3)\}$.

$$\begin{aligned}\boldsymbol{u}_1 &= \frac{(0,5,1)}{\sqrt{26}}, \quad \boldsymbol{u}_2 = \frac{(0,1,-5)}{\sqrt{26}}\\ \boldsymbol{v}_3 &= \boldsymbol{v}_3 - \mathrm{Proj}_{\boldsymbol{u}_1}\boldsymbol{v}_3 - \mathrm{Proj}_{\boldsymbol{u}_2}\boldsymbol{v}_3 = (1,-2,3) - \left(\frac{-7}{26}(0,5,1)\right) - \left(\frac{-17}{26}(0,1,-5)\right)\\ &= (1,0,0).\end{aligned}$$

Thus, an orthonormal basis is $\left\{\frac{(0,5,1)}{\sqrt{26}},\ \frac{(0,1,-5)}{\sqrt{26}},\ (1,0,0)\right\}$.

4. Find the distance from the point $(1, -2, 3)$ to the plane $2x_1 - 3x_2 + 6x_3 = 0$.

A direction normal to the plane is given by the vector $\boldsymbol{n} = (2, -3, 6)$, and $(0, 0, 0)$ is a point on the plane. The vector

$$\boldsymbol{x} = (1, -2, 3) - (0, 0, 0) = (1, -2, 3)$$

can be thought of as starting at a point on the plane and terminating at the point $(1, -2, 3)$. Thus, the projection of $\boldsymbol{x}$ onto $\boldsymbol{n}$ can be thought of as a vector starting on the plane, which is perpendicular to the plane, and terminates at the given point. So the length of this projection will give us the distance from the point to the plane.

$$\begin{aligned} \text{Proj}_{\boldsymbol{n}}\boldsymbol{x} &= \frac{2+6+18}{49}(2,-3,6) = \frac{26}{49}(2,-3,6) \\ \text{distance to plane} &= \left\|\frac{26}{49}(2,-3,6)\right\| = \frac{26}{7}. \end{aligned}$$

5. Find the distance from the point $(1, -2, 3)$ to the plane $2x_1 - 3x_2 + 6x_3 = 2$.

The computation is exactly the same as in the previous problem. Set $\boldsymbol{n} = (2, -3, 6)$ and $\boldsymbol{x} = (1, -2, 3) - (1, 0, 0) = (0, -2, 3)$.

$$\begin{aligned} \text{Proj}_{\boldsymbol{n}}\boldsymbol{x} &= \frac{24}{49}(2,-3,6) \\ \text{distance to plane} &= \left\|\frac{24}{49}(2,-3,6)\right\| = \frac{24}{7} \end{aligned}$$

9. Find an orthonormal basis for the kernels of each of the following matrices:

a. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -1 & 2 \\ 4 & 6 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & -1 & 3 \\ -3 & 1 & 0 & 1 \end{bmatrix}$

a. Matrix has rank equal to 2, so kernel has dimension 1. A basis is $\{(2, -1)\}$, and an orthonormal basis is $\left\{\frac{(2,-1)}{\sqrt{5}}\right\}$.

b. Matrix has rank equal to 2, so kernel has dimension 1. A basis is $\{(-3/2, 1/2, 1)\}$, and an orthonormal basis is $\left\{\frac{-3,1,2)}{\sqrt{14}}\right\}$.

c. Matrix has rank equal to 2, so kernel has dimension 2. A basis is $\{(1, 3, 1, 0), (-3, -10, 0, 1)\}$. Using the Gram-Schmidt procedure we construct an orthonormal basis $\left\{\frac{(1,3,1,0)}{\sqrt{11}}, \frac{(0,-1,3,1)}{\sqrt{11}}\right\}$.

10. Find an orthonormal basis for the ranges of each of the matrices in problem 9.

a. The range of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ equals S[(1,3)]. An orthonormal basis is $\{\frac{(1,3)}{\sqrt{10}}\}$.

b. and c. The ranges of the matrices in parts b. and c. are equal to $\mathbb{R}^2$. Thus, an orthonormal basis for their ranges is $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

13. Let $V = P_2$. Define $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_0^1 \boldsymbol{f}(t)\boldsymbol{g}(t)dt$. The set $B = \{1, t, t^2\}$ is a basis for V. Construct an orthonormal basis for V from B by using the Gram–Schmidt procedure.

$$\boldsymbol{u}_1 = \frac{\mathbf{1}}{\left(\int_0^1 1^2\,dt\right)^{1/2}} = \mathbf{1}$$

$$\boldsymbol{v}_2 = t - \text{Proj}_{\boldsymbol{u}_1} t = t - \frac{1}{2}$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{\left(\int_0^1 (t-1/2)^2\,dt\right)^{1/2}} = \sqrt{12}\left(t - \frac{1}{2}\right)$$

$$\boldsymbol{v}_3 = t^2 - \text{Proj}_{\boldsymbol{u}_1} t^2 - \text{Proj}_{\boldsymbol{u}_2} t^2 = t^2 - t + \frac{1}{6}$$

$$\boldsymbol{u}_3 = \frac{\boldsymbol{v}_3}{\left(\int_0^1 (t^2 - t + \frac{1}{6})^2\,dt\right)^{1/2}} = 6\sqrt{5}\left(t^2 - t + \frac{1}{6}\right)$$

16. Let $V = S[(1,0,1),(1,1,1)]$. Show that $(-1,0,1)$ is perpendicular to every vector in V.

Let $\boldsymbol{x}_1 = (1,0,1)$ and $\boldsymbol{x}_2 = (1,1,1)$. We first observe that the vector $(-1,0,1)$ is perpendicular to both of the $\boldsymbol{x}'s$.

$$\langle(-1,0,1), \boldsymbol{x}_1\rangle = -1 + 1 = 0 \quad \langle(-1,0,1), \boldsymbol{x}_2\rangle = -1 + 1 = 0\,.$$

If $\boldsymbol{x} \in V$, then there are constants c_1 and c_2 such that $\boldsymbol{x} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$. Thus,

$$\begin{aligned} \langle(-1,0,1), \boldsymbol{x}\rangle &= \langle(-1,0,1), c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2\rangle = c_1\langle(-1,0,1), \boldsymbol{x}_1\rangle + c_2\langle(-1,0,1), \boldsymbol{x}_2\rangle \\ &= 0 + 0 = 0 \end{aligned}$$

5. Find an orthonormal basis of eigenvectors for each of the following matrices:

a. $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$.

The eigenvalues are $\lambda = -1 \pm 2\sqrt{2}$. Eigenvectors associated with them are: $(1, -1+\sqrt{2})$ and $(1, -1-\sqrt{2})$ respectively. These eigenvectors are orthogonal, so to get an orthonormal basis we just need to normalize them.

$$\boldsymbol{u}_1 = \frac{(1, -1+\sqrt{2})}{\sqrt{4-2\sqrt{2}}} \qquad \boldsymbol{u}_2 = \frac{(1, -1-\sqrt{2})}{\sqrt{4+2\sqrt{2}}}$$

b. $\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$

Eigenvalues and eigenvectors are: 6 and 4, with $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively. The eigenvectors $\boldsymbol{e}_i$ are an orthonormal basis of $\mathbb{R}^2$.

c. $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

The eigenvalues and their associated eigenvectors are: $\frac{5+\sqrt{5}}{2}, (-2, -1+\sqrt{5})$ and $\frac{5-\sqrt{5}}{2}, (-2, -1-\sqrt{5})$. The eigenvectors are orthogonal so we just need to normalize them.

$$\boldsymbol{u}_1 = \frac{(-2, -1+\sqrt{5})}{\sqrt{10-2\sqrt{5}}} \quad \boldsymbol{u}_2 = \frac{(-2, -1-\sqrt{5})}{\sqrt{10+2\sqrt{5}}}$$

6. Find an orthonormal basis of eigenvectors for each of the following matrices:

a. $\begin{bmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}$

The eigenvalues and eigenvectors are:

λ	$\boldsymbol{x}_\lambda$
6	$(-1, -1, 1)$
9	$(1, 0, 1), (-1, 1, 0)$

Notice that the eigenvector associated with 6 is orthogonal to the two eigenvectors associated with the eigenvalue 9. Using the Gram-Schmidt algorithm we construct from this basis of eigenvectors an orthonormal basis of eigenvectors.

$$\boldsymbol{u}_1 = \frac{(-1,-1,1)}{\sqrt{3}}, \quad \boldsymbol{u}_2 = \frac{(1,0,1)}{\sqrt{2}}, \quad \boldsymbol{u}_3 = \frac{(-1,2,1)}{\sqrt{6}}$$

b. $\begin{bmatrix} -2 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
The eigenvalues and eigenvectors are:

λ	$\boldsymbol{x}_\lambda$
2	$(0,0,1)$
$1+3\sqrt{2}$	$(1,1+\sqrt{2},0)$
$1-3\sqrt{2}$	$(1,1-\sqrt{2},0)$

An orthogonal basis is:

$$\boldsymbol{u}_1 = (0,0,1), \quad \boldsymbol{u}_2 = \frac{(1,1+\sqrt{2},0)}{\sqrt{4+2\sqrt{2}}}, \quad \boldsymbol{u}_3 = \frac{(1,1-\sqrt{2},0)}{\sqrt{4-2\sqrt{2}}}$$

8. For each of the matrices A of problem 5 find P and D such that $P^TAP = D$, where D is a diagonal matrix.

a. $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$.
The eigenvalues are: $\lambda = -1 \pm 2\sqrt{2}$, and an orthonormal basis of eigenvectors of this matrix is $\left\{ \frac{(1,-1+\sqrt{2})}{\sqrt{4-2\sqrt{2}}}, \frac{(1,-1-\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \right\}$ respectively. Thus, we have

$$D = \begin{bmatrix} -1+2\sqrt{2} & 0 \\ 0 & -1-2\sqrt{2} \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{-1+2\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-2\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{bmatrix}.$$

b. $\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$ This matrix is already diagonal. So set D equal to it, and $P = I_2$.

c. $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. An orthornormal basis of eigenvectors of A is:

$$\boldsymbol{u}_1 = \frac{(-2,-1+\sqrt{5})}{\sqrt{10-2\sqrt{5}}} \quad \boldsymbol{u}_2 = \frac{(-2,-1-\sqrt{5})}{\sqrt{10+2\sqrt{5}}},$$

with eigenvalues $\frac{5+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$ respectively. Thus,

$$D = \begin{bmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{bmatrix}, \text{ and } P = \begin{bmatrix} \frac{-2}{\sqrt{10-2\sqrt{5}}} & \frac{-2}{\sqrt{10+2\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \end{bmatrix}$$

9. For each of the matrices A of problem 6 find P and D such that $P^T AP = D$, where D is a diagonal matrix.

a. $\begin{bmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}$

The eigenvalues and an orthonormal basis of eigenvectors are given in the table below.

λ	$\boldsymbol{x}_\lambda$
6	$\frac{(-1,-1,1)}{\sqrt{3}}$
9	$\frac{(1,0,1)}{\sqrt{2}}, \frac{(-1,2,1)}{\sqrt{6}}$

Thus, the matrices are:

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

1. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & 4 \end{bmatrix}$

a. Determine the range of A, and show that (1,1,0) is not in the range, i.e., the equation $A\boldsymbol{x} = (1,1,0)^T$ does not have a solution.

The range of A is the span of the two columns of A. To see that $\boldsymbol{x} = (1,1,0)^T$ is not in the range of A, note that the augmented matrix of this system has rank 3. That is, the rank of A and the rank of the augmented matrix are not equal, which means that the equation $A\boldsymbol{x} = (1,1,0)^T$ does not have a solution.

b. Compute $A^T A$ and show that it is one to one.

$$A^T A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 14 \\ 14 & 21 \end{bmatrix}, \quad \det\left(A^T A\right) = 77 \,.$$

Since the determinant of $A^T A$ is not zero, this matrix is one-to-one.

c. Solve the equation $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$, where $\boldsymbol{b} = (1,1,0)$.

$$A^T A\boldsymbol{x} \quad = \quad A^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The solution to this system of equations is: $\boldsymbol{x} = (0, 1/7)$.

d. If $\boldsymbol{x}$ is your solution from part c, show that $\|A\boldsymbol{x} - \boldsymbol{b}\|$ is smaller than $\|\boldsymbol{w} - \boldsymbol{b}\|$ for any vector $\boldsymbol{w}$ in the range of A.
We know that $\|\mathrm{Proj}_{Rg(A)}\boldsymbol{b} - \boldsymbol{b}\| \leq \|\boldsymbol{w} - \boldsymbol{b}\|$ for all $\boldsymbol{w} \in Rg(A)$, and since $A\boldsymbol{x} = \mathrm{Proj}_{Rg(A)}\boldsymbol{b}$, we have the desired inequality.

2. Determine the straight-line least squares fit for the following data: (1,1), $(2,-3)$, (4,0), (5,1), (10,3).

We want to find m and b such that the data satisfies the equation $y = mx + b$. This leads to the following system of equations for the unknowns m and b:

$$m + b = 1, \quad 2m + b = -3, \quad 4m + b = 0, \quad 5m + b = 1, \quad 10m + b = 3\,.$$

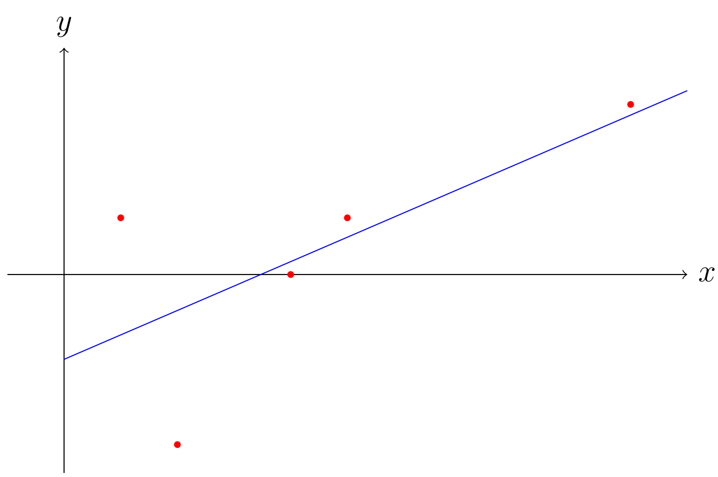
Writing this as a matrix equation we have

$$A \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Note that A has full rank, i.e., 2, which means that $A^T A$ is invertible. And the best possible values of m and b are given as solutions of

$$A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 30 \\ 2 \end{bmatrix}.$$

That is, $\begin{bmatrix} 146 & 22 \\ 22 & 5 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 30 \\ 2 \end{bmatrix}$, and $m = 53/123$, $b = -184/123$. A plot of the data points and the straight line least squares fit is shown below.



4. Consider the system of equations:

$$\begin{aligned} 3x_1 + 4x_2 + 8x_3 &= 0 \\ x_1 \qquad\quad - x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

a. This system is overdetermined (more equations than unknowns) and may not have a solution. Show that if there is a solution, it is unique.

The coefficient matrix

$$A = \begin{bmatrix} 3 & 4 & 8 \\ 1 & 0 & -1 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

has rank equal to 3. Thus, its null space is just the zero vector and hence A defines a 1-to-1 linear transformation.

b. Show that this system does not have a solution, and then find $\boldsymbol{x}$ in $\mathbb{R}^3$ such that $A\boldsymbol{x}$ is that vector in the range of A closest to (0,1,0,0).

The augmented matrix of this system has rank equal to 4, which does not equal the rank of the coefficient matrix. Thus, the system does not have a solution. To find $\boldsymbol{x}$, we only have to solve the normal equations:

$$A^T A\boldsymbol{x} = A^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 15 & 15 & 32 \\ 15 & 18 & 37 \\ 32 & 37 & 82 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} .$$

Solving this system we get $x_1 = 128/243, x_2 = 29/243, x_3 = -22/81$.

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3. Compute the inner product and the cosine of the angle between each of the following pairs of vectors:

a. $(-4, 5)$, (1,2)

$$\langle(-4,5),(1,2)\rangle = -4+10 = 6, \quad \cos\theta = \frac{6}{\sqrt{205}}.$$

b. $(-2,3,7),(2,-4,5)$

$$\langle(-2,3,7),(2,-4,5)\rangle = -4-12+35 = 19, \quad \cos\theta = \frac{19}{\sqrt{62}\sqrt{45}}.$$

c. $(-1,-2,3,5)$, (1,1,0,8)

$$\langle(-1,-2,3,5),(1,1,0,8)\rangle = -1-2+40 = 37, \quad \cos\theta = \frac{37}{\sqrt{39}\sqrt{66}}.$$