Integrals of the form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$, where $R(x,y)$ is a rational function of x and y, can be evaluated using residues:

Make substitution $z = e^{i\theta}$. Then

$$dz = e^{i\theta}\, i\, d\theta = iz\, d\theta, \quad d\theta = \frac{dz}{iz} = -i\frac{dz}{z}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$$

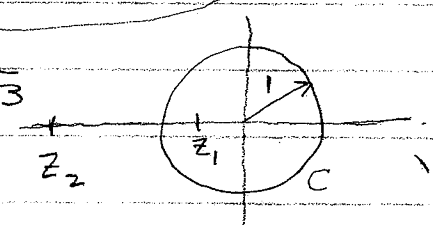
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = -i \oint_{|z|=1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{z}$$

Example. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

$$= \frac{1}{i}\oint_{|z|=1} \frac{1}{2+\frac{z+\frac{1}{z}}{2}}\,\frac{dz}{z} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+4z+1}$$

$$z_1, z_2 = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$$



$$z^2+4z+1 = (z-z_1)(z-z_2)$$

$$\operatorname{Res}\left(\frac{1}{z^2+4z+1}, z_1\right) = \left.\frac{z-z_1}{(z-z_1)(z-z_2)}\right|_{z=z_1} = \frac{1}{z_1-z_2} = \frac{1}{2\sqrt{3}}$$

$$\rightarrow \frac{\cancel{2}}{\cancel{i}}(2\pi \cancel{i})\frac{1}{\cancel{2}\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

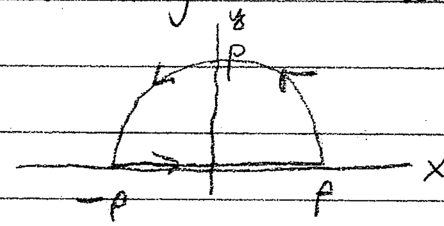
Let $p(x)$ and $q(x)$ be real polynomials such that $q(x)$ has no zeros on the x axis and $\deg p(x) \le \deg q(x) - 2$. Then the integrals

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix}\, dx$$

(whose real and imaginary parts are

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos x\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin x\, dx \ )$$

can be evaluated using the residue thm and the closed curve C_ρ



and letting $\rho \to \infty$.

<u>Example</u>. Evaluate $I = \int_0^{\infty} \frac{\cos ax}{x^2+1}\, dx \quad , \ a > 0$

Note that $2I = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\, dx$

$$= \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\, dx$$

$$\int_{C_\rho} \frac{e^{iaz}}{z^2+1}\, dz = 2\pi i \, \mathrm{Res}\left(\frac{e^{iaz}}{(z-i)(z+i)}, i \right)$$

$$= \cancel{2}\pi \cancel{i} \, \frac{e^{-a}}{\cancel{2i}} = \pi e^{-a} \quad \text{for } \rho > 1$$

So for $\rho > 1$,

$$\pi e^{-a} = \int_{C_\rho} \frac{e^{iaz}}{z^2+1}\,dz = \underbrace{\int_{-\rho}^{\rho} \frac{e^{iax}}{x^2+1}\,dx}_{J_1(\rho)} + \underbrace{\int_{C_\rho^+} \frac{e^{ia\overset{x+iy}{\widetilde{z}}}}{z^2+1}\,dz}_{J_2(\rho)}$$

$$|J_2(\rho)| \leq \pi\rho \max_{C_\rho^+} \frac{e^{-ay}}{|z|^2-1} \leq \pi\rho \frac{1}{\rho^2-1} \underset{\text{as } \rho\to\infty}{\longrightarrow} 0$$

So

$$\pi e^{-a} = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx$$

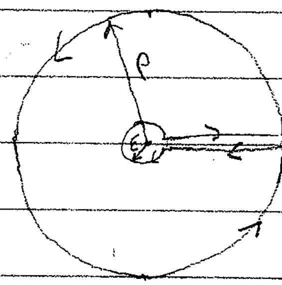
So $2I = \operatorname{Re} \pi e^{-a} = \pi e^{-a}$

$$I = \frac{\pi}{2} e^{-a}$$

Let $p(x)$ and $q(x)$ be real polynomials such that $q(x)$ has no zeros on the nonnegative x-axis and $\deg p(x) \le \deg p(x) - 2$. Let $0 < \alpha < 1$ be a constant. Then the integral

$$\int_0^\infty x^\alpha \frac{p(x)}{q(x)}\, dx$$

can be evaluated using the residue thm and the closed curve $C_{\rho,\varepsilon}$



Example. Evaluate $I = \int_0^\infty \frac{x^{1/3}}{(x+1)^2}\, dx$.

For $0 < \varepsilon < 1 < \rho$, $\int_{C_{\rho,\varepsilon}} \frac{z^{1/3}}{(z+1)^2}\, dz = 2\pi i \operatorname{Res}\left(\frac{z^{1/3}}{(z+1)^2}, -1 \right)$

$$= 2\pi i \frac{d}{dz} z^{1/3} \Big|_{z=-1} = 2\pi i \, \frac{1}{3} z^{-\frac{2}{3}} \Big|_{z=-1}$$

$$= \frac{2\pi i}{3} (-1)^{-\frac{2}{3}} = \frac{2\pi i}{3} \left(e^{\pi i}\right)^{-\frac{2}{3}} = \frac{2\pi i}{3} e^{-\frac{2}{3}\pi i}.$$

$$z = |z|\, e^{i\theta} \qquad 0<\theta<2\pi$$
$$z^{1/3} = |z|^{1/3}\, e^{\frac{i\theta}{3}}$$

So letting $f(z) = \dfrac{z^{1/3}}{(z+1)^2}$ we have

$$\frac{2\pi i}{3}\, e^{-\frac{2}{3}\pi i} = \oint_{C_\rho} f(z)\,dz - \oint_{C_\varepsilon} f(z)\,dz + \int_\varepsilon^\rho \frac{x^{1/3}}{(x+1)^2}\,dx + \int_\rho^\varepsilon \frac{x^{1/3} e^{i\frac{2\pi}{3}}}{(x+1)^2}\,dx$$

$$= \underbrace{\oint_{C_\rho} f(z)\,dz}_{\substack{\to 0 \\ \text{as } \rho\to\infty}} - \underbrace{\oint_{C_\varepsilon} f(z)\,dz}_{\substack{\to 0 \\ \text{as } \varepsilon\to 0^+}} + \left(1 - e^{i\frac{2\pi}{3}}\right)\int_\varepsilon^\rho \frac{x^{1/3}}{(x+1)^2}\,dx$$

$$\longrightarrow \left(1 - e^{i\frac{2\pi}{3}}\right)\int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx \qquad \text{as } \begin{matrix}\rho\to\infty \\ \varepsilon\to 0^+\end{matrix}$$

$$\int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx = \frac{2\pi i}{3}\, \frac{e^{-\frac{2}{3}\pi i}}{1-e^{\frac{2\pi i}{3}}} \cdot \frac{e^{-\frac{1}{3}\pi i}}{e^{-\frac{1}{3}\pi i}}$$

$$= \frac{2\pi i}{3}\, \frac{\overbrace{e^{-\pi i}}^{=-1}}{-\left(e^{\frac{\pi i}{3}} - e^{-\frac{\pi i}{3}}\right)}$$

$$= \frac{2\pi \cancel{i}}{3}\, \frac{1}{\cancel{2i}\sin\frac{\pi}{3}} = \frac{\pi}{3}\,\frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

[redacted]