Arbitrarily Large Solutions of Nonlinear Elliptic Inequalities at an Isolated Singularity

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1. Statement of Problem

We discuss the growth near the origin of C^2 positive solutions $u(x)$ of

$$af(u) \leq -\Delta u \leq f(u) \quad \text{in} \quad \mathbf{B}^n - \{0\} \tag{1}$$

where $\mathbf{B}^n = \{x \in \mathbb{R}^n \colon |x| < 1\}$, $a \in [0,1)$ is a constant, and $f\colon\ (0,\infty) \to (0,\infty)$ is a continuous function.

It seems, at least for certain f, that studying (1) is more appropriate than studying $-\Delta u = f(u)$ in $\mathbf{B}^n - \{0\}$.

Question. *Does (1) have C^2 positive solutions which are arbitrarily large near the origin?*

(i.e. for each continuous function $\varphi\colon\ (0,1) \to (0,\infty)$ does there exist a C^2 positive solution $u(x)$ of (1) such that

$$u(x) \neq \mathcal{O}(\varphi(|x|)) \quad \text{as} \quad |x| \to 0^+ ?) \tag{2}$$

This is an important question because when the answer is no, one can usually show that all C^2 positive solutions of (1) are nearly radial near the origin.

Let u be a C^2 positive solution of (1). Then $-\Delta u \geq 0$ in $\mathbf{B}^n - \{0\}$. Thus when $n = 1$

$$u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to 0^+,$$

and when $n \geq 2$

$$u(x) \in L^1\left(\frac{1}{2}\mathbf{B}^n\right).$$

Hence if $\varphi(r)$ is "large" (for example $\varphi(r) \geq \frac{1}{r^n}$) near $r = 0$ and $u(x)$ satisfies (2) then near the origin, $u(x)$ can be larger than $\varphi(|x|)$ only on a set of relatively small measure, that is

$$\lim_{\varepsilon \to 0} \frac{|\{x \in B_\varepsilon(0) : u(x) > \varphi(|x|)\}|}{|B_\varepsilon(0)|} = 0.$$

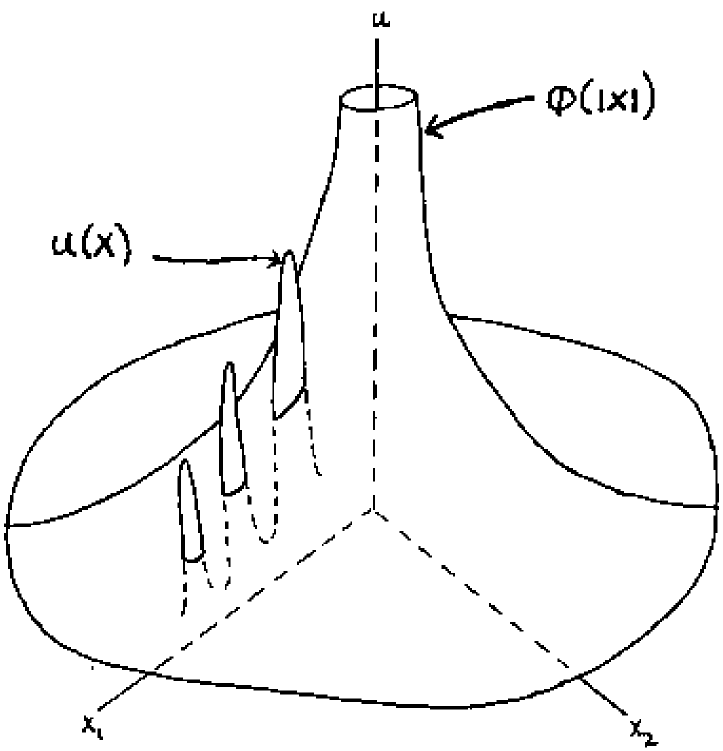

Figure 1

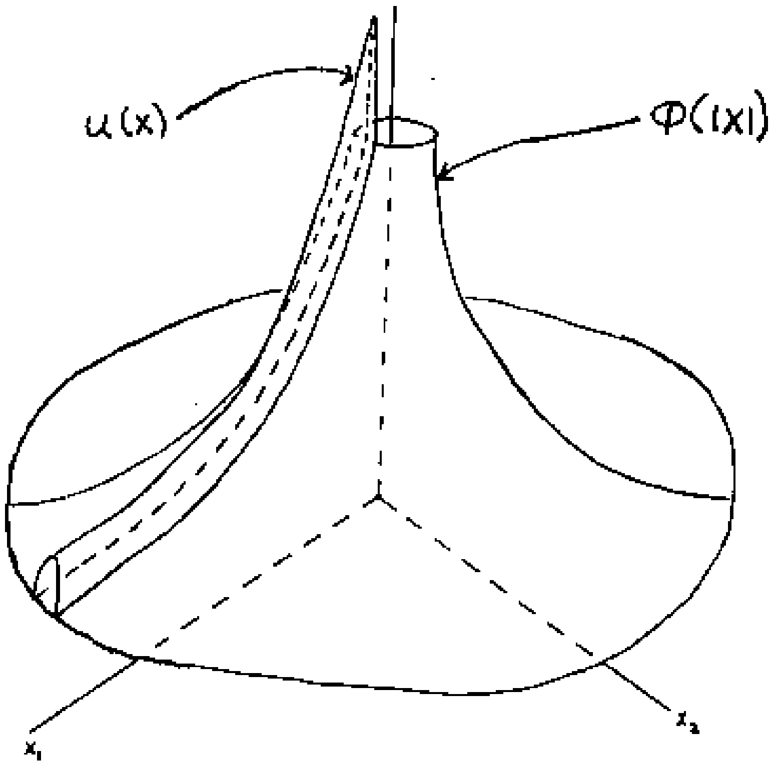

Figure 2

$$af(u) \leq -\Delta u \leq f(u) \quad \text{in} \quad \mathbf{B}^n - \{0\}$$

When $n = 2$ and $a = 0$, the function $f(t) = e^t$ is critical.

When $n \geq 3$ and $a = 0$, the function $f(t) = t^{n/(n-2)}$ is critical.

When $n \geq 3$ and $0 < a < 1$, the function $f(t) = t^{(n+2)/(n-2)}$ is critical.

2. Results when $n \geq 3$ and $a = 0$

Consider

$$0 \leq -\Delta u \leq f(u) \quad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \geq 3. \tag{1}$$

Critical growth rate for f is $f(t) \sim t^{n/(n-2)}$ as $t \to \infty$.

Theorem 1. *The problem (1), where $f\colon (0,\infty) \to (0,\infty)$ is any continuous function satisfying*

$$\lim_{t\to\infty} \frac{f(t)}{t^{n/(n-2)}} = \infty,$$

has C^2 positive solutions which are arbitrarily large near the origin.

Theorem 2. *Let $u(x)$ be a C^2 positive solution of (1) where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying*

$$f(t) = \mathcal{O}(t^{n/(n-2)}) \quad \text{as} \quad t \to \infty. \tag{2}$$

Then

$$u(x) = \mathcal{O}(|x|^{2-n}) \quad \text{as} \quad |x| \to 0^+ \tag{3}$$

and

$$0 < C_1 \leq \frac{u(x)}{\bar{u}(|x|)} \leq C_2 < \infty \quad \text{for } |x| \text{ small and positive} \tag{4}$$

where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$.

Theorem 2 is optimal in two ways. First, and more important, the growth condition (2) on $f(t)$ cannot be weakened because of Theorem 1.

Second, the conclusion (3) cannot be strengthened because $|x|^{2-n}$ is a C^2 positive solutions of (1).

Open Question. *Can (4) be strengthened to*

$$\frac{u(x)}{\bar{u}(|x|)} \to 1 \quad \text{as} \quad |x| \to 0^+.$$

3. Results when $n = 2$ and $a = 0$

Consider

$$0 \le -\Delta u \le f(u) \quad \text{in} \quad \mathbf{B}^2 - \{0\}. \tag{1}$$

Critical growth rate for f is $\log f(t) \sim t$ as $t \to \infty$.

Theorem 1. *The problem (1), where* $f\colon (0,\infty) \to (0,\infty)$ *is any continuous function satisfying*

$$\lim_{t\to\infty} \frac{\log f(t)}{t} = \infty,$$

has C^2 *positive solutions which are arbitrarily large near the origin.*

Theorem 2. *Let* $u(x)$ *be a* C^2 *positive solution of (1) where* $f\colon (0,\infty) \to (0,\infty)$ *is a continuous function satisfying*

$$\log f(t) = \mathcal{O}(t) \quad \textit{as} \quad t \to \infty. \tag{2}$$

Then

$$u(x) = \mathcal{O}\left(\log \frac{1}{|x|}\right) \quad \textit{as} \quad |x| \to 0^+. \tag{3}$$

Theorem 2 is optimal in two ways. First, and more important, the growth condition (2) on $f(t)$ cannot be weakened because of Theorem 1.

Second, the conclusion (3) cannot be strengthened because $\log \frac{1}{|x|}$ is a C^2 positive solutions of (1).

However we are not able to show that $u(x)$ in Theorem 2 satisfies

$$0 < C_1 \le \frac{u(x)}{\bar{u}(|x|)} \le C_2 < \infty \quad \text{for } |x| \text{ small and positive}$$

where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$, but we make the

Conjecture. *If u is as in Theorem 2 then either*

$$\frac{u(x)}{\log \frac{1}{|x|}} \to m \quad \text{as} \quad |x| \to 0^+ \tag{4}$$

for some positive constant m or u has a C^1 extension to the origin.

This conjecture is true if the condition on u is slightly strengthened. More precisely, if u is a C^2 positive solution in $\mathbf{B}^2 - \{0\}$ of either

$$ae^u \le -\Delta u \le e^u \qquad \text{or} \qquad 0 \le -\Delta u \le f(u),$$

where $a \in (0,1)$ is a constant and $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\log f(t) = o(t) \quad \text{as} \quad t \to \infty,$$

then either u satisfies (4) for some positive constant m or u has a C^1 extension to the origin.

Two corollaries of Theorem 2 are

Corollary 1. *Let $v(x)$ be a C^2 solution of*

$$\left.\begin{aligned} &0 \le -\Delta v \le |x|^{-a} e^v \\ &v(x) > -a \log \frac{1}{|x|} \end{aligned}\right\} \quad \textit{in} \quad \mathbf{B}^2 - \{0\}$$

where a is a positive constant. Then

$$v(x) = \mathcal{O}\left(\log \frac{1}{|x|}\right) \quad \textit{as} \quad |x| \to 0^+.$$

Proof. Let $u(x) = v(x) + a \log \frac{1}{|x|}$. Then $u(x)$ is a C^2 positive solution of

$$0 \le -\Delta u \le e^u \quad \text{in} \quad \mathbf{B}^2 - \{0\}.$$

Thus Corollary 1 follows from Theorem 2.

Corollary 2. *Let $u(x)$ be a C^2 solution of*

$$\begin{aligned} &0 \leq -\Delta u \leq |x|^a e^u \\ &u(x) > -a \log |x| \end{aligned} \qquad \textit{in} \quad \mathbb{R}^2 - \overline{\mathbf{B}^2}$$

where a is a positive constant. Then

$$u(x) = \mathcal{O}(\log |x|) \quad \textit{as} \quad |x| \to \infty.$$

Proof. Apply the Kelvin transform $u(x) = v(y)$, $x = y/|y|^2$ and then use Corollary 1.

Recall that C^2 solutions u of

$$\begin{matrix} 0 \le -\Delta u \le u^{n/(n-2)} \\ u > 0 \end{matrix} \qquad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \ge 3$$

satisfy $\quad u(x) = \mathcal{O}(|x|^{2-n}) \quad$ as $\quad |x| \to 0^+$.

However the problem

$$\begin{matrix} 0 \le -\Delta u \le u^{\lambda} \\ u > 0 \end{matrix} \qquad \text{in} \quad \mathbf{B}^n - \{0\}, \quad \frac{n}{n-2} < \lambda < \frac{n+2}{n-2}$$

has arbitrarily large solutions near the origin.

Consider instead the more restricted problem

$$\begin{matrix} au^{\lambda} \le -\Delta u \le u^{\lambda} \\ u > 0 \end{matrix} \qquad \text{in} \quad \mathbf{B}^n - \{0\}, \quad \frac{n}{n-2} < \lambda < \frac{n+2}{n-2}$$

where $0 < a < 1$.
Arbitrarily large solutions near the origin?
Answer depends on a.
Thus this is the correct problem to study for λ as above.

4. Results when $n \geq 3$, $0 < a < 1$ and $f(t) = t^\lambda$, $\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}$.

Consider

$$au^\lambda \leq -\Delta u \leq u^\lambda \quad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \geq 3 \tag{1}$$

where

$$\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}. \tag{2}$$

Theorem 1. *Suppose λ satisfies (2). Then there exists $a = a(n, \lambda) \in (0,1)$ such that (1) has C^2 positive solutions which are arbitrarily large near the origin.*

Theorem 2. *Suppose λ satisfies (2). Then there exists $a = a(n, \lambda) \in (0,1)$ such that if u is a C^2 positive solution of (1) then*

$$u(x) = \mathcal{O}(|x|^{-2/(\lambda-1)}) \quad \textit{as} \quad |x| \to 0^+$$

and

$$0 < C_1 \leq \frac{u(x)}{\bar{u}(|x|)} \leq C_2 < \infty \quad \textit{for } |x| \textit{ small and positive}$$

where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$.

Theorem 1 is not true when $\lambda \leq n/(n-2)$ by a previous result.

Let λ satisfy (2) and let

$$I_1 = I_1(n, \lambda) = \{a \in (0,1)\colon \text{Theorem 1 is true}\}$$

$$I_2 = I_2(n, \lambda) = \{a \in (0,1)\colon \text{Theorem 2 is true}\}.$$

Then I_1 and I_2 are nonempty disjoint subintervals of $(0,1)$.

Open Question. *Does* $I_1 \cup I_2 = (0,1)$*? If not, what is the behavior of* C^2 *positive solutions of (1) when*

$$a \in (0,1) - (I_1 \cup I_2)?$$

5. Results when $n \geq 3$ and $f(t) = t^{\frac{n+2}{n-2}}$.

For sharper results consider instead of

$$au^{\frac{n+2}{n-2}} \leq -\Delta u \leq u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \geq 3$$

the problem

$$k(x)u^{\frac{n+2}{n-2}} \leq -\Delta u \leq u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \geq 3. \tag{1}$$

where $k\colon \mathbf{B}^n - \{0\} \to (0,1]$ is continuous.

Theorem (T and Lei Zhang). *A necessary and sufficient condition for (1) to have C^2 positive solutions which are arbitrarily large near the origin is that k be less than one on a sequence of points in $\mathbf{B}^n - \{0\}$ which tends to the origin.*

Necessity proved by Caffarelli, Gidas, and Spruck.

6. Results when $n \geq 3$ and $f(t) = t^\lambda$, $\lambda > \frac{n+2}{n-2}$.

Theorem. *Suppose $\lambda > \frac{n+2}{n-2}$. Then there exists $a = a(n,\lambda) \in (0,1)$ such that*

$$au^\lambda \leq -\Delta u \leq u^\lambda \quad \textit{in} \quad \mathbf{B}^n - \{0\}, \quad n \geq 3 \tag{1}$$

has C^2 positive solutions which are arbitrarily large near the origin. Moreover, if $\lambda > \frac{(n-1)+2}{(n-1)-2}$ then, for each $a \in (0,1)$, problem (1) has C^2 positive solutions which are arbitrarily large near the origin.

Conjecture. *The problem*

$$-\Delta u = u^\lambda \quad \textit{in} \quad \mathbf{B}^n - \{0\}, \quad \textit{where} \quad n \geq 3,$$

has C^2 positive solutions which are arbitrarily large near the origin iff $\lambda > \frac{(n-1)+2}{(n-1)-2}$

7. Global Existence

Consider

$$au^\lambda \le -\Delta u \le u^\lambda \quad \text{in} \quad \mathbb{R}^n, \quad n \ge 3 \tag{1}$$

where

$$\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}. \tag{2}$$

Theorem 1. *Suppose λ satisfies (2). Then there exists $a = a(n,\lambda) \in (0,1)$ such that there exists a C^2 positive solution $u(x)$ of (1).*

Theorem 2. *Suppose λ satisfies (2). Then there exists $a = a(n,\lambda) \in (0,1)$ such that there does not exist a C^2 positive solution $u(x)$ of (1).*

Consider the

Question. *For what values of $\lambda \in \mathbb{R}$ and $a \in (0,1]$ does there exist a C^2 positive solution of (1)?*

To answer this question we consider three mutually exclusive possibilities for the value of λ.

Case I. Suppose $-\infty < \lambda \le n/(n-2)$. Then for each $a \in (0,1]$, (1) does not have a C^2 positive solution because Serrin and Zou prove that if $0 \le \lambda \le n/(n-2)$ then there does not exist a C^2 positive solution of $-\Delta u \ge u^\lambda$ in $\mathbb{R}^n - \mathbf{B}^n$ and this result can be easily extended to include all negative values of λ.

Case II. Suppose λ satisfies (2). Then our Theorems 1 and 2 hold.

Case III. Suppose $(n+2)/(n-2) \le \lambda < \infty$. Then for each $a \in (0,1]$, (1) has a C^2 positive solution because Fowler shows that (1) with $a = 1$ has a C^2 positive radial solution.