Existence of Large Singular Solutions of Conformal Scalar Curvature Equations in S^n

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Consider positive solutions u of the scalar curvature equation

$$\underbrace{-\Delta u+\frac{n(n-2)}{4}u}_{Lu}=Ku^{\frac{n+2}{n-2}}\qquad\text{in}\qquad S^n \tag{1}$$

where K is a positive function in $C^1(S^n)$.

Definition. *The singular set $\mathcal{S}_u$ of a weak positive solution u of (1) is the set of all $Q\in S^n$ such that u is unbounded in every neighborhood of Q.*

Elliptic theory $\quad\Longrightarrow\quad$ u is C^2 in $S^n-\mathcal{S}_u$.
In particular, if $\mathcal{S}_u=\{Q\}$, then u is C^2 in $S^n-\{Q\}$.

Theorem A [CGS]. *Equation (1) with $K\equiv 1$ in S^n does not have a weak positive solution whose singular set consists of a single point.*

Theorem B [Schoen]. *Equation (1) with $K\equiv 1$ in S^n has a weak positive solution u whose singular set is any prescribed finite subset of S^n consisting of at least two points.*

Theorem C [CGS]. *If u is a C^2 positive solution of $Lu=u^{\frac{n+2}{n-2}}$ in a punctured neighborhood of some point $Q\in S^n$ then*

$$u(P)=O\left(|P-Q|^{\frac{-(n-2)}{2}}\right)\qquad\text{as}\qquad P\to Q.$$

Our main result is

Theorem 1. *Let $\varphi\colon(0,1)\to(0,\infty)$ be any large continuous function. Then every positive function κ in $C^1(S^n)$, $n\geq 6$, can be approximated in the $C^1(S^n)$ norm by a positive function $K\in C^1(S^n)$ such that for some $Q\in S^n$ there exists a weak positive solution u of (1) with singular set $\{Q\}$ satisfying*

$$u(P)\neq O(\varphi(|P-Q|))\qquad\text{as}\qquad P\to Q.$$

Remark 1. Theorem 1 is not true when n is 3 [Chen and Lin] or 4 [T, Zhang]. *Open question*: Is Theorem 1 true when $n = 5$?

Remark 2. When $\kappa \equiv 1$ in S^n, $n \geq 3$, the analog of Theorem 1 concerning the approximation of κ in the $C^0(S^n)$ norm instead of the $C^1(S^n)$ norm is true [T, Zhang] and [Leung].

Remark 3. By a result of C.S. Lin, the point Q in Theorem 1 must be a critical point of K.

To prove Theorem 1, choose $Q \in S^n$ such that $\nabla\kappa(Q) = 0$. Under the stereographic projection of S^n onto $\mathbf{R}^n \cup \{\infty\}$ taking Q to the origin in $\mathbf{R}^n$, we see that Theorem 1 is implied by the following theorem concerning the equation

$$-\Delta u = K(x) u^{\frac{n+2}{n-2}} \qquad \text{in} \qquad \mathbf{R}^n - \{0\}, \quad n \geq 6. \tag{2}$$

Theorem 2. *Let $\varphi\colon (0,1) \to (0,\infty)$ be any large continuous function and let $\varepsilon > 0$. Suppose $\kappa\colon \mathbf{R}^n \to \mathbf{R}$ is a C^1 function which is bounded between positive constants and satisfies $\nabla\kappa(0) = 0$. Then there exits a C^1 function $K\colon \mathbf{R}^n \to \mathbf{R}$ satisfying $\|K - \kappa\|_{C^1(\mathbf{R}^n)} < \varepsilon$ and $K(x) = \kappa(x)$ for $|x| \geq \varepsilon$ such that (2) has a C^2 positive solution $u(x)$ satisfying*

$$u(x) = O(|x|^{2-n}) \qquad \text{as} \quad |x| \to \infty$$

and

$$u(x) \neq O(\varphi(|x|) \quad \text{as} \quad |x| \to 0^+.$$

Idea of proof of Theorem 2. By scaling, we can assume $\kappa(0) = 1$. Since $\nabla\kappa(0) = 0$, we can assume $\kappa \equiv 1$ in some neighborhood of the origin, because if $\hat{\kappa}$ is related to κ as shown, then

$$\|\hat{\kappa} - \kappa\|_{C^1(\mathbf{R}^n)} < \varepsilon/2$$

provided $\delta > 0$ is
sufficiently small.

$$\text{Let} \quad w_\sigma(r) = [n(n-2)]^{\frac{n-2}{4}} \left(\frac{\sigma}{\sigma^2 + r^2}\right)^{\frac{n-2}{2}}. \quad \text{Then}$$

$$-\Delta w_\sigma(|x|) = w_\sigma(|x|)^{\frac{n+2}{n-2}} \qquad \text{for } x \in \mathbf{R}^n \text{ and } \sigma > 0$$

and

$$\begin{matrix} w_\sigma(0) \to \infty \\ w_\sigma(r) \to 0, \quad r \neq 0 \end{matrix} \qquad \text{as} \quad \sigma \to 0^+.$$

These functions $w_\sigma(|x|)$ are sometimes called bubbles.

Let $B_{2r_i}(x_i)$ be a sequence of disjoint balls in $\mathbf{R}^n - \{0\}$ such that $x_i \to 0$ and $r_i \to 0$.

Let $u_i(x) = w_{\sigma_i}(|x - x_i|)$ be a bubble centered at x_i. Then, provided σ_i tends to 0 sufficiently fast, we have

$$u_i(x_i) = w_{\sigma_i}(0) > \varphi(|x_i|),$$

$$\hat{u}(x) := \sum_{i=1}^{\infty} u_i(x) \in C^{\infty}(\mathbf{R}^n - \{0\})$$

and

$$-\Delta \hat{u} \sim \hat{u}^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{R}^n - \{0\}.$$

We try to find $u_0 \colon (\mathbf{R}^n - \{0\}) \to (0,1)$ such that

$$u := u_0 + \hat{u} \qquad \text{and} \qquad K := \frac{-\Delta u}{u^{\frac{n+2}{n-2}}}$$

satisfy the conclusion of Theorem 2. The function u_0 will be obtained as a positive solution of

$$-\Delta u_0 = H(x, u_0) \qquad \text{in} \qquad \mathbf{R}^n - \{0\} \tag{3}$$

for some appropiate function $H \colon \mathbf{R}^n \times [0, \infty) \to (0, \infty)$. We will use the method of sub and super-solutions to solve (3).

H is chosen as follows: First define a $C^1(\mathbf{R}^n)$ function k by

$$k(x) = \begin{cases} \kappa(x) - \varepsilon_i, & \text{if } x \in B_{r_i}(x_i) \\ \kappa(x), & \text{if } x \in \mathbf{R}^n - \bigcup_{i=1}^{\infty} B_{2r_i}(x_i) \end{cases}$$

where $\varepsilon_i \to 0$.

To force K close to κ, we require

$$
\begin{aligned}
& k \le K \le \kappa \quad \text{in} \quad \mathbf{R}^n - \{0\} \\
\Longleftrightarrow \quad & k \le \frac{-\Delta u}{u^{\frac{n+2}{n-2}}} \le \kappa \quad \text{in} \quad \mathbf{R}^n - \{0\} \\
\Longleftrightarrow \quad & k(x) \le \frac{H(x, u_0(x)) + \sum_{i=1}^{\infty} u_i(x)^{\frac{n+2}{n-2}}}{\left(u_0(x) + \sum_{i=1}^{\infty} u_i(x)\right)^{\frac{n+2}{n-2}}} \le \kappa(x), \quad x \in \mathbf{R}^n - \{0\} \\
\Longleftrightarrow \quad & \underline{H}(x, u_0(x)) \le H(x, u_0(x)) \le \overline{H}(x, u_0(x)), \quad x \in \mathbf{R}^n - \{0\}
\end{aligned}
$$

where

$$
\underline{H}(x, v) := k(x) \left(v + \sum_{i=1}^{\infty} u_i(x)\right)^{\frac{n+2}{n-2}} - \sum_{i=1}^{\infty} u_i(x)^{\frac{n+2}{n-2}}
$$

$$
\overline{H}(x, v) := \kappa(x) \left(v + \sum_{i=1}^{\infty} u_i(x)\right)^{\frac{n+2}{n-2}} - \sum_{i=1}^{\infty} u_i(x)^{\frac{n+2}{n-2}}
$$

for $x \in \mathbf{R}^n$ and $v \ge 0$. So H will be chosen so that

$$
\underline{H}(x, v) \le H(x, v) \le \overline{H}(x, v) \quad \text{for} \quad (x, v) \in \mathbf{R}^n \times [0, \infty).
$$

Then we check, when $n \ge 6$, that $\|K - \kappa\|_{C^1(\mathbf{R}^n)} < \varepsilon$.