# Isolated Singularities of Nonlinear Elliptic Inequalities

Steven D. Taliaferro

Mathematics Department
Texas A&M University
College Station, TX 77843-3368
stalia@math.tamu.edu

## 1. Introduction

It is well-known that if $u$ is positive and harmonic in a punctured neighborhood of the origin in $\mathbf{R}^n$ ($n \geq 2$) then either $u$ has a $C^2$ extension to the origin, or for some finite positive number $m$,

$$\lim_{x \to 0} \frac{u(x)}{\Phi(|x|)} = m,$$

where

$$\Phi(|x|) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x|}, & \text{if } n = 2 \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & \text{if } n \geq 3 \end{cases}$$

is the fundamental solution of $-\Delta$. In particular, $u$ satisfies the following two weaker conditions:

(i) $u$ is *asymtotically radial* as $x \to 0$, i.e.

$$\lim_{x \to 0} \frac{u(x)}{\bar{u}(|x|)} = 1,$$

where $\bar{u}(r)$ is the average of $u$ on the sphere $|x| = r$, and

(ii) $u(x) = O(\Phi(|x|)) \quad \text{as} \quad x \to 0.$

Do similar results hold for $C^2$ positive solutions $u$ of the differential inequalities

$$0 \le -\Delta u \le f(u) \qquad \text{in} \qquad \mathbf{B}^n - \{0\}, \tag{1.1}$$

where $f\colon (0,\infty) \to (0,\infty)$ is a given continuous function? Here $\mathbf{B}^n$ is a ball in $\mathbf{R}^n$ centered at the origin whose radius depends on the solution $u$.

Specifically, under what conditions on the function $f$ does every $C^2$ positive solution $u$ of (1.1) satisfy some (or all) of the following three conditions?

(i) $u$ is asymptotically radial as $x \to 0$,
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$,
(iii) $u$ is *asymptotically harmonic* as $x \to 0$, i.e.

$$\lim_{x\to 0} \frac{u(x)}{h(x)} = 1$$

for some function $h(x)$ which is positive and harmonic in a punctured neighborhood of the origin in $\mathbf{R}^n$.

Since (iii) implies (i) and (ii), the conditions on $f$ for (iii) to hold will have to be at least as strong as the conditions on $f$ for (i) or (ii) to hold.

## 2. Two dimensional results

**Theorem 2.1.** *Let $u(x)$ be a $C^2$ positive solution of*

$$0 \le -\Delta u \le f(u) \qquad \text{in} \qquad \mathbf{B}^2 - \{0\},$$

*where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying*

$$\log f(t) = O(t) \quad \text{as} \quad t \to \infty.$$

*Then $u$ is asymptotically harmonic as $x \to 0$.*

The conformal Gauss curvature equation

$$-\Delta u = e^u$$

has been extensively studied. A corollary of the previous theorem is the following.

**Corollary.** *If $u(x)$ is a $C^2$ positive solution of*

$$0 \le -\Delta u \le e^u \qquad \text{in} \qquad \mathbf{B}^2 - \{0\}$$

*then $u$ is asymptotically harmonic as $x \to 0$.*

The condition on $f$ in the previous theorem was

$$\log f(t) = O(t) \quad \text{as} \quad t \to \infty. \tag{2.1}$$

The following theorem shows that this condition on $f$ is essentially optimal.

**Theorem 2.2.** *Let $f\colon(0,\infty)\to(0,\infty)$ and $\varphi\colon(0,1)\to(0,\infty)$ be continuous functions such that*

$$\lim_{t\to\infty}\frac{\log f(t)}{t} = \infty \quad \textit{and} \quad \lim_{t\to 0^+}\varphi(t) = \infty.$$

*Then there exists a $C^2$ positive solution $u$ of*

$$0 \le -\Delta u \le f(u) \qquad \textit{in} \qquad \mathbf{B}^2 - \{0\}$$

*such that*

$$u(x) \ne O(\varphi(|x|) \quad \textit{as} \quad x \to 0$$

*and $u$ is not asymptotically radial as $x \to 0$.*

Thus condition (2.1) is essentially optimal for any (or all) of the following conditions to hold:

(i) $u$ is asymptotically radial as $x \to 0$,
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$,
(iii) $u$ is asymptotically harmonic as $x \to 0$.

There is no analogous condition on $f$ in three and higher dimensions.

## 3. Three and higher dimensional results

**Theorem 3.1.** *Let $u(x)$ be a $C^2$ positive solution of*

$$0 \leq -\Delta u \leq f(u) \qquad \textit{in} \qquad \mathbf{B}^n - \{0\}, \quad n \geq 3,$$

*where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying*

$$f(t) = O(t^{\frac{n}{n-2}}) \quad \textit{as} \quad t \to \infty.$$

*Then $u$ is asymptotically radial as $x \to 0$. Moreover, either $u$ is asymptotically harmonic as $x \to 0$ or $u$ satisfies the following two conditions:*

$$\lim_{x\to 0} \frac{u(x)}{\Phi(|x|)} = 0$$

*and*

$$\liminf_{x\to 0} \frac{u(x)}{\left(\frac{\Phi(|x|)}{(\log \Phi(|x|))^{(n-2)/2}}\right)} > 0.$$

The condition on $f$ in the previous theorem was

$$f(t) = O(t^{\frac{n}{n-2}}) \quad \text{as} \quad t \to \infty. \tag{3.1}$$

The following theorem shows that this condition on $f$ is essentially optimal.

**Theorem 3.2.** *Let $f\colon (0,\infty) \to (0,\infty)$ and $\varphi\colon (0,1) \to (0,\infty)$ be continuous functions such that*

$$\lim_{t\to\infty} \frac{f(t)}{t^{n/(n-2)}} = \infty \quad \text{and} \quad \lim_{t\to 0^+} \varphi(t) = \infty.$$

*Then there exists a $C^2$ positive solution $u$ of*

$$0 \le -\Delta u \le f(u) \qquad \text{in} \qquad \mathbf{B}^n - \{0\}, \quad n \ge 3,$$

*such that*

$$u(x) \ne O(\varphi(|x|) \quad \text{as} \quad x \to 0$$

*and $u$ is not asymptotically radial as $x \to 0$.*

Thus condition (3.1) is essentially optimal for either (or both) of the following conditions to hold:

(i) $u$ is asymptotically radial as $x \to 0$,
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$;

but is too weak to imply
(iii) $u$ is asymptotically harmonic as $x \to 0$,

because for $0 < \sigma < (n-2)/2$ the function

$$u_\sigma(x) := \frac{\Phi(|x|)}{(\log \Phi(|x|))^\sigma}$$

is a $C^2$ positive solution of $0 \le -\Delta u \le u^{\frac{n}{n-2}}$ in $\mathbf{B}^n - \{0\}$ and $u_\sigma(x)$ is not asymptotically harmonic as $x \to 0$.

This is in contrast to the situation in two dimensions.

## 4. Asymptotically harmonic solutions in three and higher dimensions

By the last two theorems, the essentially optimal growth condition on $f$ for $u$ to be asymptotically radial as $x \to 0$ is

$$f(t) = O(t^{n/(n-2)}) \quad \text{as} \quad t \to \infty.$$

In the following theorem, we strenghen this growth condition on $f$ in such a way as to conclude that $u$ is asymptotically harmonic as $x \to 0$.

First we need a definition:
$\log_1 := \log, \quad \log_2 := \log \circ \log, \quad \log_3 := \log \circ \log \circ \log, \quad$ etc.

**Theorem 4.1.** *Let $u$ be a $C^2$ positive solution of*

$$0 \le -\Delta u \le \frac{u^{\frac{n}{n-2}}}{(\log_1 u) \cdots (\log_{q-1} u)(\log_q u)^\beta} \tag{4.1}$$

*in $\mathbf{B}^n - \{0\}$, $n \ge 3$, where $\beta \in (1, \infty)$ and $q$ is a positive integer. Then $u$ is asymptotically harmonic as $x \to 0$.*

This theorem is essentially optimal because a solution of (4.1) when $\beta = 1$ is

$$u(|x|) = \frac{\Phi(|x|)}{\log_{q+2} \Phi(|x|)}$$

which is not asymptotically harmonic as $x \to 0$.

## 5. Summary of results

In summary, the essentially optimal condition on a continuous function $f\colon(0,\infty)\to(0,\infty)$ for every $C^2$ positive solution $u$ of

$$0\le -\Delta u\le f(u) \qquad \text{in} \qquad \mathbf{B}^n-\{0\}$$

to satisfy

(i) $u$ is asymptotically radial as $x\to 0$, and/or
(ii) $u(x)=O(\Phi(|x|))$ as $x\to 0$

is

$$\log f(t)=O(t) \quad \text{when} \quad n=2,$$

$$f(t)=O(t^{n/(n-2)}) \quad \text{when} \quad n\ge 3.$$

Moreover, the essentially optimal condition on $f$ for $u$ to be asymptotically harmonic as $x\to 0$ is

$$\log f(t)=O(t) \quad \text{when} \quad n=2,$$

$$f(t)=O\left(\frac{t^{n/(n-2)}}{(\log_1 t)\cdots(\log_{q-1} t)(\log_q t)^\beta}\right) \quad \text{when} \quad n\ge 3$$

for some $\beta\in(1,\infty)$ and some positive integer $q$.

## 6. Further results

Recall that $C^2$ solutions $u$ of

$$\begin{array}{c} 0 \le -\Delta u \le u^{n/(n-2)} \\ u > 0 \end{array} \qquad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \ge 3$$

satisfy $\quad u(x) = O(|x|^{2-n}) \quad$ as $\quad x \to 0.$

However the problem

$$\begin{array}{c} 0 \le -\Delta u \le u^{\lambda} \\ u > 0 \end{array} \qquad \text{in} \quad \mathbf{B}^n - \{0\}, \quad \frac{n}{n-2} < \lambda$$

has arbitrarily large solutions near the origin. (That is given a continuous function $\varphi\colon (0,1) \to (0,\infty)$ there exists a $C^2$ solution $u$ such that $u(x) \ne O(\varphi(|x|))$ as $x \to 0$.)

Consider instead the more restricted problem

$$\begin{array}{c} au^{\lambda} \le -\Delta u \le u^{\lambda} \\ u > 0 \end{array} \qquad \text{in} \quad \mathbf{B}^n - \{0\}, \quad \frac{n}{n-2} < \lambda < \frac{n+2}{n-2}$$

where $0 < a < 1$.
Arbitrarily large solutions near the origin?
Answer depends on $a$.
Thus this is the correct problem to study for $\lambda$ as above.

More precisely, consider the differential inequalities

$$au^\lambda \le -\Delta u \le u^\lambda \quad \text{in} \quad \mathbf{B}^n - \{0\}, \quad n \ge 3 \tag{1}$$

where

$$\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}. \tag{2}$$

**Theorem 1.** *Suppose $\lambda$ satisfies (2). Then there exists $a = a(n,\lambda) \in (0,1)$ such that (1) has $C^2$ positive solutions which are arbitrarily large near the origin.*

**Theorem 2.** *Suppose $\lambda$ satisfies (2). Then there exists $a = a(n,\lambda) \in (0,1)$ such that if $u$ is a $C^2$ positive solution of (1) then*

$$u(x) = O(|x|^{-2/(\lambda-1)}) \quad \textit{as} \quad x \to 0$$

*and*

$$0 < C_1 \le \frac{u(x)}{\bar{u}(|x|)} \le C_2 < \infty \quad \textit{for } |x| \textit{ small and positive}$$

*where $\bar{u}(r)$ is the average of $u$ on the sphere $|x| = r$.*

Let $\lambda$ satisfy (2) and let

$$I_1 = I_1(n,\lambda) = \{a \in (0,1)\colon \text{Theorem 1 is true}\}$$

$$I_2 = I_2(n,\lambda) = \{a \in (0,1)\colon \text{Theorem 2 is true}\}.$$

Then $I_1$ and $I_2$ are nonempty disjoint subintervals of $(0,1)$.

**Open Question.** *Does $I_1 \cup I_2 = (0,1)$? If not, what is the behavior of $C^2$ positive solutions of (1) when*

$$a \in (0,1) - (I_1 \cup I_2)?$$

## 7. The critical exponent $\lambda = \frac{n+2}{n-2}$

For every $a \in (0,1)$ the inequalities

$$au^{\frac{n+2}{n-2}} \leq -\Delta u \leq u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{B}^n - \{0\}$$

have $C^2$ positive solutions which are arbitrarily large near the origin. (Here and later $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$). In fact we have the following stronger result concerning the inequalities

$$k(x)u^{\frac{n+2}{n-2}} \leq -\Delta u \leq u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{B}^n - \{0\} \tag{7.1}$$

where $k \colon \mathbf{B}^n \to (0,1]$ is continuous and $k(0) = 1$.

**Theorem 7.1.** *A necessary and sufficient condition for (7.1) to have $C^2$ positive solutions which are arbitrarily large near the origin is that $k$ be less than one on a sequence of points in $\mathbf{B}^n - \{0\}$ which tends to the origin.*

Necessity proved by Caffarelli, Gidas, and Spruck.
Sufficiency proved by T, Lei Zhang, and M.C. Leung.

**Corollary 7.2.** *Let $\varphi \colon (0,1) \to (0,\infty)$ be a continuous function. Then the function $\kappa(x) \equiv 1$ can be approximated in the $C^0(\mathbf{B}^n)$ norm by a function $K \in C^0(\mathbf{B}^n)$ such that the PDE*

$$-\Delta u = K(x)u^{\frac{n+2}{n-2}} \quad \textit{in} \quad \mathbf{B}^n - \{0\}$$

*has a $C^2$ positive solution satisfying*

$$u(x) \neq O(\varphi(|x|)) \quad \textit{as} \quad x \to 0.$$

**Proof**. In Theorem 7.1 take $K := \frac{-\Delta u}{u^{\frac{n+2}{n-2}}}$.

A sharper result is the following:

**Theorem 7.3.** *Let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function. Then the function $\kappa(x) \equiv 1$ can be approximated in the $C^1(\mathbf{B}^n)$ norm by a function $K \in C^1(\mathbf{B}^n)$ such that the PDE*

$$-\Delta u = K(x) u^{\frac{n+2}{n-2}} \quad \textit{in} \quad \mathbf{B}^n - \{0\}, \quad n \geq 6,$$

*has a $C^2$ positive solution satisfying*

$$u(x) \neq O(\varphi(|x|)) \quad \textit{as} \quad x \to 0.$$

This theorem is not true if $n = 3$ (C.S. Lin) or if $n = 4$ (T, Lei Zhang), because in these dimensions

$$u(x) = O(|x|^{-(n-2)/2}) \quad \text{as} \quad x \to 0.$$