Sample Questions for Small Exam 2  Math 314, Spring 2002, March 12

1. Let $S$ be the set

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(a) Show that $S$ spans the space $\mathbb{R}^3$ and find a basis for $\mathbb{R}^3$ contained in $S$.

(b) What are the coordinates of $\vec{c} = (1, 1, 1)$ with respect to the basis found in (a)?

2. Let $A$ be a matrix of size $m \times n$ and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^m$ such that the system $A\vec{x} = \vec{b}$ has a unique solution and the system $A\vec{x} = \vec{c}$ has no solution. Explain carefully and precisely why $m > n$ must hold.

3. Show that none of the following sets of vectors is a subspace of $\mathbb{R}^3$:

   (a) The set $U$ of vectors $\vec{x} = (x_1, x_2, x_3)$ such that $x_1^2 + x_2^2 = 1$.

   (b) The set $V$ of vectors in $\mathbb{R}^3$ whose all three coordinates (components) are integers.

   (c) The set $W$ of vectors in $\mathbb{R}^3$ that have at least one coordinate (component) equal to 0.

4. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 2 & -2 \\ 1 & 2 & c \end{bmatrix},$$

where $c$ is some real constant.

(a) Find the unique value of $c$ for which the columns of $A$ do not form a basis of $\mathbb{R}^3$.

(b) For the value of $c$ found in (a), calculate the dimension and a basis for the nullspace of $A$.

(c) For the value of $c$ found in (a), calculate the dimension and a basis for the column space of $A$.

5. Let $M_{2,2}$ be the vector space of all $2 \times 2$ matrices

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and define a transformation a function $F : M_{2,2} \to M_{2,2}$ by

$$F(X) = AX.$$

(a) Show that $F$ is a linear transformation.

(b) Determine the dimension of $\ker(F)$ and a basis for $\ker(F)$.

6. I solved a certain homogeneous linear system of 4 equations in 5 unknowns and noticed that all solutions are scalar multiples of a single non-zero vector. Then I changed the right hand sides of the equations to get a nonhomogeneous system. Is this new system necessarily consistent? Why?

7. Determine a basis of $\mathbb{R}^3$ that includes the vector $\vec{u}_1 = (2, 1, 1)$. 

8. (15 points) The entries of a $3 \times 3$ "checkerboard" matrix $A$ are defined by

$$a_{ij} = 1 \quad \text{if } i + j \text{ is even}$$

and

$$a_{ij} = 0 \quad \text{if } i + j \text{ is odd.}$$

Find the dimensions and bases for the column space and the nullspace of $A$. 