1. Consider the following linear transformation from $P_3$ to $P_4$:

$$L(p(x)) = \int_0^x p(t) \, dt$$

(a) Find a basis for the kernel of $L$.

(b) Find a basis for the range of $L$.

Answer:

First, we will find the matrix representing $L$ with respect to the basis $[1, x, x^2]$.

$$L(1) = \int_0^x 1 \, dt = \left[t\right]_0^x = x$$
$$L(x) = \int_0^x t \, dt = \left[\frac{1}{2} t^2\right]_0^x = \frac{1}{2} x^2$$
$$L(x^2) = \int_0^x t^2 \, dt = \left[\frac{1}{3} t^3\right]_0^x = \frac{1}{3} x^3$$

Thus, the matrix representing $L$ with respect to the basis $[1, x, x^2]$ is

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

(a) The nullspace of $A$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; thus, the kernel of $L$ is the zero polynomial $p(x) = 0$. 

1
(b) The columnspace of $A$ is the span of
\[
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Thus, the range of $L$ is $\text{Span}(x, x^2, x^3)$.

2. For each of the following, determine whether the functions $f(x)$, $g(x)$ and $h(x)$ are linearly dependent or linearly independent in $C[-\pi, \pi]$. Explain your answers.

(a) $f(x) = \sin x$
   
   $g(x) = \cos x$

   $h(x) = \sin 2x$

(b) $f(x) = e^x + e^{-x}$
   
   $g(x) = e^x - e^{-x}$

   $h(x) = e^x$

Answer:

(a) Suppose that there exists $c_1, c_2, c_3$ such that

\[ c_1 \sin x + c_2 \cos x + c_3 \sin 2x = 0 \]

If $x = 0$, we get the equation

\[ c_2 = 0 \]

If $x = \frac{\pi}{2}$, we get the equation

\[ c_1 = 0 \]

If $x = \frac{\pi}{4}$, we get the equation

\[ \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}} + c_3 = 0 \]

Since, $c_1 = c_2 = 0$ from above, this equation gives us $c_3 = 0$.

Thus, $c_1 = c_2 = c_3 = 0$, so the functions are linearly independent.
(b) Note that \( f(x) + g(x) = (e^x + e^{-x}) + (e^x - e^{-x}) = 2e^x = 2h(x) \).

Thus, \( f(x) + g(x) - 2h(x) = 0 \), so the functions are not linearly dependent.

3. Consider the following subspace of \( P_3 \):

\[
S = \left\{ p(x) \in P_3 \mid p(2) - p(1) = 0 \right\}
\]

Find a basis for this subspace.

**Answer:** Suppose that \( p(x) = ax^2 + bx + c \) is a polynomial in \( S \). Then, \( p(2) = 4a + 2b + c \) and \( p(1) = a + b + c \), so that \( p(2) - p(1) = 3a + b \). Thus, \( 3a + b = 0 \), so \( b = -3a \). Thus, we can write \( p(x) \) as

\[
p(x) = ax^2 - 3ax + c = a(x^2 - 3x) + c
\]

Thus, every polynomial in \( S \) is in the span of the polynomials \( x^2 - 3x \) and \( 1 \). Since these polynomials are linearly independent, they form a basis for \( S \). Thus, a basis for \( S \) is \( \{x^2 - 3x, 1\} \).

4. Suppose that \( E = [u_1, u_2] \) is a basis for \( R^2 \). Then, \( F = [u_1 + u_2, u_1 - u_2] \) is also a basis for \( R^2 \). Find the transition matrix from \( F \) to \( E \).

**Answer:** The transition matrix from \( F \) to \( E \) is

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\]

5. Suppose that \( A \) is a 3 \( \times \) 3 matrix with eigenvalues 1, 0, -1. What are the eigenvalues of the matrix \( A - 42I \), where \( I \) is the 3 \( \times \) 3 identity matrix?

**Answer:** Let \( B = A - 42I \). We want to find \( \lambda \) such that \( B - \lambda I \) is singular. Since the eigenvalues of \( A \) are 1, 0, -1, we know that \( A - I \), \( A \), and \( A + I \) are singular.

Thus, \( B + 42I = A - 42I + 42I = A \) is singular.

Similarly, \( B + 43I = A - 42I + 43I = A + I \) is singular.

And, \( B + 41I = A - 42I + 41I = A - I \) is singular.
Thus, the eigenvalues of $B$ are $-42, -43, -41$.

6. Consider the following matrix:

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

Compute $e^A$.

**Answer:** First, we compute the eigenvalues and eigenvectors of $A$. Since $A$ is upper triangular, the entries along the diagonal are the eigenvalues. Thus, $A$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$.

For $\lambda_1 = 2$:

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, the eigenvector corresponding to $\lambda_1 = 2$ is $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 3$:

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} = \text{Span} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Thus, the eigenvector corresponding to $\lambda_2 = 3$ is $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Since $A$ has 2 distinct eigenvalues, $A$ is diagonalizable. We can factor $A$ as $A = XD X^{-1}$ where

$$X = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$
This allows us to compute $e^A$, since $e^A = Xe^DX^{-1}$:

$$e^A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^2 & 2e^2 - 2e^3 \\ 0 & e^3 \end{pmatrix}$$

7. Solve the differential equation $y'' = 4y + 3y'$ with initial conditions $y(0) = 3$ and $y'(0) = 2$.

**Answer:** By setting $y_1 = y$ and $y_2 = y'$, we can turn this second order differential equation in a system of first order differential equations:

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The eigenvalues for this matrix are $\lambda_1 = 4$ and $\lambda_2 = -1$ and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus, the general solution to the system is

$$y_1 = c_1e^{4t} + c_2e^{-t}$$
$$y_2 = 4c_1e^{4t} - c_2e^{-t}$$

The initial conditions give us the equations $c_1 + c_2 = 3$ and $4c_1 - c_2 = 2$. Solving for $c_1$ and $c_2$, we get $c_1 = 1$ and $c_2 = 2$. Thus, the solution to the original second order differential equation with the given initial conditions is

$$y(t) = e^{4t} + 2e^{-t}$$
8. (a) Find the distance from the point \((1, 1, 1)\) to the plane \(2x + 2y + z = 0\).
(b) Find the distance from the point \((1, 2)\) to the line \(4x - 3y = 0\).

Answer:

(a) The vector \(\mathbf{N} = (2, 2, 1)^T\) is perpendicular to the plane, and the point \((0, 0, 0)\) is on the plane.
Consider the point \((1, 1, 1)\); call it \(P\). And consider the point \((0, 0, 0)\); call it \(Q\) (this is the point on the plane). Consider the vector \(\overrightarrow{QP}\). If we project \(\overrightarrow{QP}\) onto \(\mathbf{N}\), we get a vector whose distance is the distance from the plane to \(P\).

\[
\overrightarrow{QP} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

The scalar projection of \(\overrightarrow{QP}\) onto \(\mathbf{N}\) is

\[
\alpha = \frac{\overrightarrow{QP} \cdot \mathbf{N}}{\|\mathbf{N}\|} = \frac{5}{3}
\]

Thus, the distance between the point and the plane is \(\frac{5}{3}\).

(b) To find the point on the line \(4x - 3y = 0\) that is closest to the point \((1, 2)\), we project the vector \(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\) onto the line \(4x - 3y = 0\).

A vector in the direction of the line \(4x - 3y = 0\) is \(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\). The projection of \(\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\) onto \(\mathbf{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\) is

\[
\mathbf{p} = \frac{\mathbf{v}^T \mathbf{w}}{\mathbf{w}^T \mathbf{w}} \mathbf{w} = \frac{11}{25} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 33/25 \\ 44/25 \end{pmatrix} = \begin{pmatrix} 1.32 \\ 1.76 \end{pmatrix}
\]

Thus, the point \((1.32, 1.76)\) is the point on the line \(4x - 3y = 0\) that is closest to the point \((1, 2)\).
9. Suppose that $S$ is the subspace of $\mathbb{R}^3$ spanned by the vectors \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \). Find a basis for $S^\perp$.

**Answer:** $S^\perp$ is the set of all vectors perpendicular to \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \). Thus, $S^\perp$ is the set of all vectors \( \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \) such that $y_1 + 2y_2 + y_3 = 0$ and $y_1 - y_2 + 2y_3 = 0$. Thus, $S^\perp$ is the nullspace of the following matrix:

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & -1 & 2
\end{pmatrix}
\]

We can row reduce this matrix:

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & -1 & 2
\end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 5 \\ 0 & -3 & 1 \end{pmatrix}
\]

Thus,

nullspace \( \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \) = Span \( \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \)

Thus, a basis for $S^\perp$ is \( \left\{ \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \right\} \).
10. Given the following table of data points

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

find the best least squares fit by a linear function $f(x) = c_1 + c_2x$.

**Answer:** We would like to find a linear function of the form $y = ax + b$ that approximates the data set. Plugging in the values of the data set into the linear function, we get

\[
\begin{align*}
1 &= -a + b \\
3 &= a + b \\
3 &= 2a + b
\end{align*}
\]

This system of equations has no solutions.

We want to find the least squares solution to the system $Ax = b$ with

\[
A = \begin{pmatrix}
-1 & 1 \\
1 & 1 \\
2 & 1
\end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix}
1 \\
3 \\
3
\end{pmatrix}.
\]

The least squares solutions are the solutions to $A^T A \hat{x} = A^T b$. First, we compute $A^T A$ and $A^T b$:

\[
A^T A = \begin{pmatrix}
6 & 2 \\
2 & 3
\end{pmatrix}
\quad \quad
A^T b = \begin{pmatrix}
8 \\
7
\end{pmatrix}
\]

Now, we can solve $A^T A \hat{x} = A^T b$:

\[
\begin{pmatrix}
6 & 2 & 8 \\
2 & 3 & 7
\end{pmatrix} \rightarrow \begin{pmatrix}
7 & 0 & 5 \\
0 & 7 & 13
\end{pmatrix}
\]

Thus, $\hat{x} = \begin{pmatrix}
5/7 \\
13/7
\end{pmatrix}$.

Thus, the best least squares linear fit to the data is $y = \frac{5}{7}x + \frac{13}{7}$.
11. Consider the following function:

\[ f(x) = \begin{cases} 
1 & 0 \leq x \leq \pi \\
0 & -\pi \leq x < 0 
\end{cases} \]

Find the best least squares approximation to \( f(x) \) on \([-\pi, \pi]\) by a trigonometric polynomial of degree less than or equal to 2.

**Answer:** We compute the coefficients of the trigonometric polynomial that approximates \( f(x) \):

\[
a_0 = \left\langle f(x), \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2}} \, dx \\
= \frac{1}{\pi} \int_{0}^{\pi} \frac{1}{\sqrt{2}} \, dx = \frac{1}{\sqrt{2}}
\]

\[
a_n = \langle f(x), \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \\
= \frac{1}{\pi} \int_{0}^{\pi} \cos(nx) \, dx \\
= \frac{1}{\pi} \left[ \frac{1}{n} \sin(nx) \right]_{0}^{\pi} = 0
\]

Thus, \( a_1 = 0 \) and \( a_2 = 0 \).
\[ b_n = \langle f(x), \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} \sin(nx) \, dx \]
\[ = \frac{1}{\pi} \left[ -\frac{1}{n} \cos(nx) \right]_0^{\pi} \]
\[ = \frac{1}{\pi} \left( -\frac{1}{n} \cos(n\pi) + \frac{1}{n} \cos(0) \right) \]
\[ = \begin{cases} 
\frac{2}{\pi n} & n \text{ odd} \\
0 & n \text{ even} 
\end{cases} \]

Thus, \( b_1 = \frac{2}{\pi} \) and \( b_2 = 0 \).

Thus, the best least squares approximation to \( f(x) \) on \([-\pi, \pi]\) by a trigonometric polynomial of degree less than or equal to 2 is

\[ f(x) \approx \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \right) + \frac{2}{\pi} \sin x = \frac{1}{2} + \frac{2}{\pi} \sin x \]

12. The set

\[ S = \{ \cos x, \sin x, \cos 2x, \sin 2x \} \]

is an orthonormal set in \( C[-\pi, \pi] \) with inner product \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx \).

Determine the value of each of the following.

(a) \( \langle \cos x, 2 \cos x - 3 \sin 2x \rangle \)

(b) \( \langle 2 \sin x - 3 \cos 2x, 3 \cos x + 3 \sin x + 4 \cos 2x \rangle \)

(c) \( \| \cos x + 3 \sin x - 2 \sin 2x + \cos 2x \| \)
Answer: Since $S$ is an orthonormal set in the inner product space, we can compute the inner products by finding the coordinates for each function with respect to the basis $S$, and then using the normal dot product.

(a) $\langle \cos x, 2 \cos x - 3 \sin 2x \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ -3 \end{pmatrix} = 2$

(b) $\langle 2 \sin x - 3 \cos 2x, 3 \cos x + 3 \sin x + 4 \cos 2x \rangle = \begin{pmatrix} 0 \\ 2 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 4 \\ 0 \end{pmatrix}
= 6 - 12 = -6$

(c) Since $S$ is an orthonormal set, we can compute the norm by finding the coordinates for the function with respect to the basis $S$, and then using the normal norm.

$$\| \cos x + 3 \sin x - 2 \sin 2x + \cos 2x \| = \sqrt{1 + 9 + 4 + 1} = \sqrt{15}$$

13. Given the basis $\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} \right\}$ for $R^3$, use the Gram-Schmidt process to obtain an orthonormal basis.

Answer: We start by computing the norm of the first vector:

$$\left\| \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\| = \sqrt{4 + 4 + 1} = 3$$

Thus, the first vector in the orthonormal basis is:
We note that the vector \( \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \) is orthogonal to \( \mathbf{u}_1 \), so all we need to do is make the second vector have unit length. We compute its norm:

\[
\left\| \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\| = \sqrt{4 + 1 + 4} = 3
\]

Thus, the second vector in the orthonormal basis is:

\[
\mathbf{u}_2 = \begin{pmatrix} -2 \\ -\frac{2}{3} \\ 1 \\ 2 \end{pmatrix}
\]

The third vector is not orthogonal to either \( \mathbf{u}_1 \) or \( \mathbf{u}_2 \). Thus, we need to use the third vector to find a vector orthogonal to \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \). We start by computing the projection of the third vector onto the plane spanned by \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \):
\[
p_2 = \left< \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \right> u_1 + \left< \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix} \right> u_2
\]
\[
= \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \\ -8 \end{pmatrix}
\]
\[
= \begin{pmatrix} 16 \\ 4 \\ -4 \end{pmatrix}
\]

Next, we compute \( \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - p_2 \):
\[
\begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 16 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}
\]

The resulting vector, \( \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} \) is orthogonal to \( u_1 \) and \( u_2 \). To make an orthonormal basis, we need to change this vector into unit length. We compute its norm:
\[
\left\| \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} \right\| = \sqrt{4 + 16 + 16} = \sqrt{36} = 6
\]

Thus,
\[ u_3 = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \]

The orthonormal basis is

\[
\begin{pmatrix}
\frac{2}{3} \\
2 \\
1
\end{pmatrix}
, \quad
\begin{pmatrix}
-\frac{2}{3} \\
1 \\
\frac{2}{3}
\end{pmatrix}
, \quad
\begin{pmatrix}
\frac{1}{3} \\
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}
\]

14. The plane \( x_1 + 2x_2 + x_3 = 0 \) is a subspace of \( \mathbb{R}^3 \). Find an orthonormal basis for this subspace.

**Answer:** We start by finding a basis for the subspace. Any two linearly independent vectors which lie on the plane will form a basis. One way to find a basis is to solve the equation \( x_1 + 2x_2 + x_3 = 0 \):

\[ x_1 = -2x_2 - x_3 \]

Thus, for any \( \alpha \) and \( \beta \) the following is a solution:

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\]

Thus, the vectors form \( \mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \) and \( \mathbf{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \) form a basis for the subspace.
To find an orthonormal basis, we use the Gram-Schmidt algorithm on this basis:

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{x_1}{\sqrt{4 + 1}} = \left( \begin{array}{c} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{array} \right)$$

The projection of $x_2$ onto $u_1$ is

$$p_1 = \langle x_2, u_1 \rangle u_1 = \frac{2}{\sqrt{5}} \left( \begin{array}{c} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{array} \right) = \left( \begin{array}{c} -\frac{4}{5} \\ \frac{2}{5} \\ 0 \end{array} \right)$$

Then, $x_2 - p_1$ is orthogonal to $u_1$.

$$x_2 - p_1 = \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) - \left( \begin{array}{c} -\frac{4}{5} \\ \frac{2}{5} \\ 0 \end{array} \right) = \left( \begin{array}{c} -\frac{1}{5} \\ \frac{2}{5} \\ 1 \end{array} \right)$$

We compute the magnitude of $x_2 - p_1$:

$$\|x_2 - p_1\| = \sqrt{\frac{1}{25} + \frac{4}{25} + 1} = \frac{\sqrt{30}}{5}$$

Thus,

$$u_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|} = \frac{5}{\sqrt{30}} \left( \begin{array}{c} -\frac{1}{5} \\ -\frac{2}{5} \\ 1 \end{array} \right) = \left( \begin{array}{c} -\frac{1}{\sqrt{30}} \\ -\frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{array} \right)$$
Thus, an orthonormal basis for the subspace is

\[
\left\{ \begin{pmatrix} -\frac{2}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{30}} \\ -\frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{pmatrix} \right\}
\]

Note: There are infinitely many possible orthonormal bases for the plane. To check that the answer you have is a correct answer, just check that the two vectors are both in the plane (by checking that they satisfy the equation for the plane \(x_1 + 2x_2 + x_3 = 0\)), that each vector has unit length, and that the dot product of the two vectors is 0.

15. Consider the linear transformation

\[L(\mathbf{x}) = \text{the projection of } \mathbf{x} \text{ onto the vector } \begin{pmatrix} 1 \\ 2 \end{pmatrix}\]

(a) Find the matrix \(A\) representing \(L\) with respect to the standard basis.

(b) Find the matrix \(B\) representing \(L\) with respect to the basis \([\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}]\).

(c) Find the matrix \(S\) such that \(B = S^{-1}AS\).

\textbf{Answer:}

(a) We start by applying \(L\) to each of the standard basis vectors:

\[
L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix}
\]

\[
L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \end{pmatrix}
\]
These vectors become the columns of $A$. Thus, the matrix representing $L$ with respect to the standard basis is

$$A = \begin{pmatrix}
\frac{1}{5} & \frac{2}{5} \\
\frac{2}{5} & \frac{4}{5}
\end{pmatrix}$$

(b) We apply $L$ to each of the basis vectors:

$$L\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{5}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$L\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, we change each of these vectors into the basis $E = \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right]$:

$$\begin{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{pmatrix}_E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}_E = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These vectors become the columns of $B$. Thus, the matrix representing $L$ with respect to the basis $E$ is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) The matrix $S$ such that $B = S^{-1}AS$ is the transition matrix from the basis $E$ to the standard basis. Thus,

$$S = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$
16. Let \( \mathbf{u}_1, \mathbf{u}_2, \) and \( \mathbf{u}_3 \) form an orthonormal basis for \( \mathbb{R}^3 \), and let \( \mathbf{u} \) be a unit vector in \( \mathbb{R}^3 \). If \( \mathbf{u}^T \mathbf{u}_1 = \frac{1}{3} \) and \( \mathbf{u}^T \mathbf{u}_2 = \frac{2}{3} \), determine the value of \( |\mathbf{u}^T \mathbf{u}_3| \).

**Answer:** Let \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) be \( \mathbf{u} \) in the basis \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \). Since this basis is orthonormal, and since \( \mathbf{u}^T \mathbf{u}_1 = \frac{1}{3} \) and \( \mathbf{u}^T \mathbf{u}_2 = \frac{2}{3} \), we know that \( a = \frac{1}{3} \) and \( b = \frac{2}{3} \). Since \( \mathbf{u} \) is a unit vector, we know that \( \sqrt{a^2 + b^2 + c^2} = 1 \). Thus,

\[
\sqrt{\frac{1}{9} + \frac{4}{9} + c^2} = 1
\]

Solving, we get \( c^2 = \frac{4}{9} \). Thus, \( |\mathbf{u}^T \mathbf{u}_3| = |c| = \frac{2}{3} \).