Solutions to Practice Test 1

1. For each of the following augmented matrices, solve the corresponding system of linear equations.

(a) \[
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 3 & -1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 1 & 4
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
1 & 2 & 1 & 5 & 1 \\
2 & 4 & 1 & 7 & 0 \\
2 & 4 & -1 & 1 & 4
\end{pmatrix}
\]

Answer: To solve each of these systems, we row reduce.

(a) \[
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]

Thus, we get the solutions

\[
x_1 = -2 \\
x_2 = -1 \\
x_3 = 2
\]

(b) \[
\begin{pmatrix}
1 & 2 & 3 & -1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 1 & 4
\end{pmatrix}
\]

The last equation is now 0 = 3. Thus, the system is inconsistent and has no solutions.
(c) \[
\begin{pmatrix}
1 & 2 & 1 & 5 & 1 \\
2 & 4 & 1 & 7 & 0 \\
2 & 4 & -1 & 1 & 4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 1 & 5 & 1 \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 8
\end{pmatrix}
\]

The last equation is now \(0 = 8\). Thus, the system is inconsistent and has no solutions.

2. Consider the following graph:

(a) Determine the adjacency matrix \(A\) of the graph.
(b) Compute \(A^2\) and \(A^3\).
(c) How many walks of length less than or equal to 3 are there from \(V_3\) to \(V_4\).

**Answer:**

(a) \(A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}\)
(b) \(A^2 = \begin{pmatrix}
2 & 1 & 1 & 2 \\
1 & 3 & 2 & 1 \\
1 & 2 & 3 & 1 \\
2 & 1 & 1 & 2
\end{pmatrix}\)
(c) \(A^3 = \begin{pmatrix}
2 & 5 & 5 & 2 \\
5 & 4 & 5 & 5 \\
5 & 5 & 4 & 5 \\
2 & 5 & 5 & 2
\end{pmatrix}\)

(c) By considering entry 3, 4 of \(A\), \(A^2\), and \(A^3\), we see that there is one walk of length 1 from \(V_3\) to \(V_4\), one walk of length 2 from \(V_3\) to \(V_4\), and one walk of length 3 from \(V_3\) to \(V_4\).
to \( V_4 \), and five walks of length 3 from \( V_3 \) to \( V_4 \). Thus, there are 
\( 1 + 1 + 5 = 7 \) walks of length less than or equal to 3 from \( V_3 \) to \( V_4 \).

3. For the following 3 matrices, determine if the matrix is invertible. If the matrix is invertible, compute the inverse.

(a) \( A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \)

(b) \( B = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \)

(c) \( C = \begin{pmatrix} 1 & 2 & 1 & -2 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 3 & 7 & 3 & 5 & 2 \end{pmatrix} \)

Answer: For each of these, we will start by computing the determinant. If the determinant if zero, then the matrix is not invertible. If the determinant is nonzero, then we will have to use some method (either row reducing \((A | I)\) or computing the adjoint) to to find the inverse.

(a) \( \det(A) = 8 - 3 = 5 \neq 0 \) Thus, \( A \) is invertible. We can compute the inverse, using the method for computing the inverse of a \( 2 \times 2 \) matrix.

\[
A = \frac{1}{5} \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{pmatrix}
\]

(b) We will compute the determinant of \( B \) by expanding along the first column.

\[
\det(B) = 1 \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = (3 - 1) - (-2 + 3) = 1 \neq 0
\]

Thus, \( B \) is invertible. We will compute the inverse by row reducing the augmented matrix \((B | I)\).
\[
\begin{pmatrix}
1 & 2 & -3 \\
0 & 1 & -1 \\
-1 & -1 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Thus,
\[
B^{-1} = \begin{pmatrix}
2 & -3 & 1 \\
1 & 0 & 1 \\
1 & -1 & 1
\end{pmatrix}
\]

(c) We start computing the determinant by expanding along row 4:
\[
\text{det}(C) = 2
\begin{vmatrix}
1 & 2 & 1 & 0 \\
0 & 0 & 3 & 0 \\
1 & 2 & 0 & 0 \\
3 & 7 & 3 & 2
\end{vmatrix}
\]

Next, we expand along row 2.
\[
\text{det}(C) = 2(-3)
\begin{vmatrix}
1 & 2 & 0 \\
1 & 2 & 0 \\
3 & 7 & 2
\end{vmatrix}
\]

Next, we expand along the third column:
\[
\text{det}(C) = 2(-3)(2)
\begin{vmatrix}
1 & 2 \\
1 & 2
\end{vmatrix} = 0
\]

Since \(\text{det}(C) = 0\), \(C\) is not invertible.

4. Write the following matrix as the product of elementary matrices.
\[
A = \begin{pmatrix}
1 & 1 & 3 \\
0 & 1 & 3 \\
-1 & -1 & -2
\end{pmatrix}
\]

4
Answer:

First, we row reduce $A$.

$$
\begin{pmatrix}
1 & 1 & 3 \\
0 & 1 & 3 \\
-1 & -1 & -2 \\
\end{pmatrix}
+ \text{row } 1 \rightarrow
\begin{pmatrix}
1 & 1 & 3 \\
0 & 1 & 3 \\
0 & 0 & 1 \\
\end{pmatrix}
- \text{row } 2
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 3 \\
0 & 0 & 1 \\
\end{pmatrix}
- 3 \cdot \text{row } 3
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

This allows us to find four elementary matrices $E_1, E_2, E_3$ such that $E_3E_2E_1A = I$:

$$
E_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix}
$$

$$
E_2 = \begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

$$
E_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -3 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

So, we have that $E_3E_2E_1A = I$. We can multiply by the inverses of these elementary matrices, and we get that $A = E_1^{-1}E_2^{-1}E_3^{-1}$. The inverses of the elementary matrices are the following:

$$
E_1^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1 \\
\end{pmatrix}
$$

$$
E_2^{-1} = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

$$
E_3^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 3 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

Thus, we can write $A$ as a product of elementary matrices:
\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{pmatrix}
\]

5. For each of the following, determine whether the given set is a subspace of \( \mathbb{R}^3 \). You do not need to explain your answer.

(a) \( S_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid 2x_1 + 3x_2 - x_3 = 0 \right\} \)

(b) \( S_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid 2x_1 + x_3 = 5 \right\} \)

(c) \( S_3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1^2 = 0 \right\} \)

(d) \( S_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 = 2x_2 = x_3 \right\} \)

(e) \( S_5 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \)

Answer:

(a) Yes, \( S_1 \) is a subspace.

(b) No, \( S_2 \) is not a subspace.

(c) Yes, \( S_3 \) is a subspace.

Comment #1: \( S_3 \) is a subspace, because \( x_1^2 = 0 \) means that \( x_1 = 0 \).
I had intended to have a subset similar to one of the following:
\[ S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 \cdot x_2 = 0 \right\} \]

or

\[ S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_2^2 - x_3^2 = 0 \right\} \]

Both of these would not be subspaces.

Comment #2: Previously, in the above comment, I claimed that the following was not a subspace.

\[ S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_2^2 + x_3^2 = 0 \right\} \]

This is a subspace, because if \( x_2^2 + x_3^2 = 0 \), then both \( x_1 = 0 \) and \( x_3 = 0 \).

Sorry, for any confusion this has caused.

(d) Yes, \( S_4 \) is a subspace.

(e) Yes, \( S_5 \) is a subspace.

6. Let \( u = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \end{pmatrix} \), \( v = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -3 \end{pmatrix} \), \( w = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 5 \end{pmatrix} \), \( x = \begin{pmatrix} 6 \\ 2 \\ 0 \\ 4 \end{pmatrix} \).

Determine whether the following sets of vectors are linearly independent or linearly dependent. Explain your answers.

(a) \( \{u, v, w\} \)

(b) \( \{u, v\} \)
(c) \{ \mathbf{u}, \mathbf{x} \}

(d) \{ \mathbf{u}, \mathbf{0} \}

**Answer:**

(a) These vectors are linearly dependent. Here is why:

Consider \( c_1, c_2, \) and \( c_3 \) such that

\[
\begin{pmatrix}
3 \\
1 \\
0 \\
2
\end{pmatrix}
+ c_2
\begin{pmatrix}
1 \\
1 \\
2 \\
-3
\end{pmatrix}
+ c_3
\begin{pmatrix}
2 \\
0 \\
-2 \\
5
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

We can write this as an augmented matrix and row reduce:

\[
\begin{pmatrix}
3 & 1 & 2 & 0 \\
1 & 1 & 0 & 0 \\
0 & 2 & -2 & 0 \\
2 & -3 & 5 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Thus, we get that there are infinitely many solutions, and \( c_1 = -c_3 \)
and \( c_2 = c_3 \). So, for example \( \mathbf{u} - \mathbf{v} - \mathbf{w} = \mathbf{0} \), and the vectors are linearly dependent.

(b) These vectors are linearly independent, because they are not multiples of each other.

(c) These vectors are linearly dependent, because \( 2\mathbf{u} = \mathbf{x} \).

(d) These vectors are linearly dependent, because any set of vectors containing \( \mathbf{0} \) is linearly dependent.

7. Consider the vector space \( P_3 \) of all polynomials with degree less than 3. Find a basis of the following subspace of \( P_3 \):

\[
S = \left\{ p(x) \in P_3 \middle| \int_0^1 p(x) \, dx = 0 \right\}
\]

**Answer:** A polynomial in \( P_3 \) is of the form \( p(x) = a + bx + cx^2 \). If the polynomial is in the subspace \( S \), then \( \int_0^1 p(x) \, dx = 0 \), which means
that \[ ax + \frac{1}{2} bx^2 + \frac{1}{3} cx^3 \] 
\[ a + \frac{b}{2} + \frac{c}{3} = 0. \]
If we solve this equation for \( a \), we get \( a = -\frac{b}{2} - \frac{c}{3} \).

Thus, every polynomial in \( S \) can be written in the form
\[ p(x) = \left( -\frac{b}{2} - \frac{c}{3} \right) + bx + cx^2. \]
We can rewrite this as
\[ p(x) = b \left( \frac{1}{2} + x \right) + c \left( \frac{1}{3} + x^2 \right) \]

Thus, every polynomial in \( S \) can be written as a linear combination of
the polynomials \(-\frac{1}{2} + x\) and \(-\frac{1}{3} + x^2\). This means that
\[ \text{Span} \left( -\frac{1}{2} + x, -\frac{1}{3} + x^2 \right) = S \]

All we need to do now is show that these two polynomials are linearly independent:

Suppose there exists \( c_1 \) and \( c_2 \) such that
\[ c_1 \left( -\frac{1}{2} + x \right) + c_2 \left( -\frac{1}{3} + x^2 \right) = 0 \]
Rearranging, we get
\[ \left( -\frac{1}{2} c_1 - \frac{1}{3} c_2 \right) + c_1 x + c_2 x^2 = 0 \]

Since this must be true for all values of \( x \), the coefficients on the left side must all be 0. Thus, using the coefficients of the \( x \) and \( x^2 \) terms, we get that \( c_1 = 0 \) and \( c_2 = 0 \).
Thus, the polynomials $-\frac{1}{2} + x$ and $-\frac{1}{3} + x^2$ are linearly independent.

We have shown that the polynomials $-\frac{1}{2} + x$ and $-\frac{1}{3} + x^2$ span $S$ and are linearly independent. Thus, they form a basis for $S$.

A basis for the subspace $S$ is $\left\{ -\frac{1}{2} + x, -\frac{1}{3} + x^2 \right\}$.

8. Let $u_1 = \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right)$ and $u_2 = \left( \begin{array}{c} -\frac{1}{4} \\ \frac{1}{4} \end{array} \right)$. Find the coordinates of the vector $\left( \begin{array}{c} 5 \\ -2 \end{array} \right)$ with respect to the basis $[u_1, u_2]$.

**Answer:** The transition matrix $U$ from the basis $[u_1, u_2]$ to the standard basis is $U = \left( \begin{array}{cc} \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{4} \end{array} \right)$. The transition matrix from the standard basis to the basis $[u_1, u_2]$ is $U^{-1}$, which we can compute using the method of computing the inverse of a $2 \times 2$ matrix.

$$U^{-1} = \frac{1}{6} \left( \begin{array}{cc} 4 & 1 \\ -2 & 1 \end{array} \right)$$

To find the coordinates of the vector $\left( \begin{array}{c} 5 \\ -2 \end{array} \right)$ with respect to the basis $[u_1, u_2]$ we just multiply by $U^{-1}$.

$$\left[ \left( \begin{array}{c} 5 \\ -2 \end{array} \right) \right]_{[u_1, u_2]} = U^{-1} \left( \begin{array}{c} 5 \\ -2 \end{array} \right) = \left( \begin{array}{c} 3 \\ -2 \end{array} \right)$$