1. Consider the following linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$:

$$L(x) = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

(a) Determine the kernel of $L$.
(b) Determine the range of $L$ (that is, the image of $\mathbb{R}^4$).

**Answer:**

(a) First we row reduce the matrix.

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, we have the solutions

$$x_1 = -x_3 - 2x_4$$
$$x_2 = -2x_3 - 3x_4$$

Letting $x_3 = s$ and $x_4 = t$, we have the solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s - 2t \\ -2s - 3t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Thus,

$$\ker(L) = \text{Span} \left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$
(b) The range of $L$ is the span of the column vectors of the matrix. Looking at the row reduced version of the matrix in part (a), we see that any three of the column vectors are linearly dependent, but any two are linearly independent (as they are not multiples of each other).

Thus,

$$\text{range}(L) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \right\}$$

2. Let $P$ be the vector space of all polynomials. Determine whether the following are linear transformations from $P$ to $P$. Explain your answers.

(a) $L(p) = xp(x)$

(b) $L(p) = x^2 + p(x)$

(c) $L(p) = \int_0^x p(t) \, dt$

Answer:

(a) $L$ is a linear transformation. To show this, we need to show that $L(\alpha p) = \alpha L(p)$ and $L(p + q) = L(p) + L(q)$:

$$L(\alpha p(x)) = x\alpha p(x)$$

$$\alpha L(p(x)) = \alpha xp(x)$$

Since $x\alpha p(x) = \alpha xp(x)$, we get that $L(\alpha p(x)) = \alpha L(p(x))$

$$L(p(x) + q(x)) = x(p(x) + q(x))$$

$$L(p(x)) + L(q(x)) = xp(x) + xq(x)$$

Since $x(p(x) + q(x)) = xp(x) + xq(x)$, we get that $L(p(x) + q(x)) = L(p(x)) + L(q(x))$.

Thus, $L$ is a linear transformation.
(b) $L$ is not a linear transformation. It fails to satisfy both $L(\alpha p) = \alpha L(p)$ and $L(p + q) = L(p) + L(q)$ (and just showing that it fails one of these suffices to show that $L$ is not a linear transformation.)

$L (\alpha p(x)) = x^2 + \alpha p(x) \\
\alpha L (p(x)) = \alpha (x^2 + p(x))$

Since $x^2 + \alpha p(x) \neq \alpha (x^2 + p(x))$ (unless $\alpha = 1$ or $p(x)$ is the zero polynomial), we get that $L$ is not a linear transformation.

$L (p(x) + q(x)) = x^2 + p(x) + q(x) \\
L (p(x)) + L (q(x)) = x^2 + p(x) + x^2 + q(x) = 2x^2 + p(x) + q(x)$

Since $x^2 + p(x) + q(x) \neq 2x^2 + p(x) + q(x)$, we get that $L$ is not a linear transformation.

(c) $L$ is a linear transformation. To show this, we need to show that $L(\alpha p) = \alpha L(p)$ and $L(p + q) = L(p) + L(q)$:

$L (\alpha p(x)) = \int_0^x \alpha p(t) \, dt \\
\alpha L (p(x)) = \alpha \int_0^x p(t) \, dt$

Since $\int_0^x \alpha p(t) \, dt = \alpha \int_0^x p(t) \, dt$ by basic properties of integration, we get that $L (\alpha p(x)) = \alpha L (p(x))$

$L (p(x) + q(x)) = \int_0^x p(t) + q(t) \, dt \\
L (p(x)) + L (q(x)) = \int_0^x p(t) \, dt + \int_0^x q(t) \, dt$

Since $\int_0^x p(t) + q(t) \, dt = \int_0^x p(t) \, dt + \int_0^x q(t) \, dt$ by basic properties of integration, we get that $L (p(x) + q(x)) = L (p(x)) + L (q(x))$.

Thus, $L$ is a linear transformation.
3. Consider the subspace $S$ of $C[0,1]$ spanned by $\cosh x$ and $\sinh x$, and consider the following two bases of the subspace $S$:

$$E = [\cosh x, \sinh x]$$

$$F = [e^x, e^{-x}]$$

(a) Find the transition matrix from $E$ to $F$.

(b) Let $D$ be the differentiation operator on $S$; that is, $D(f(x)) = f'(x)$. Find the matrix $A$ representing $D$ with respect to the basis $E$.

(c) Find the matrix $B$ representing $D$ with respect to the basis $F$.

**Answer:**

(a) We know that

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

$$\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

Thus, the transition matrix is

$$\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}$$

The first column of this matrix comes from the coordinates of $\cosh x$ in the basis $F$, and the second column of this matrix comes from the coordinates of $\sinh x$ in the basis $F$.

Here is a different way to think about it that results in the same answer:
We want to find a matrix $T$ that takes a vector $\left( \begin{array}{c} a \\ b \end{array} \right)$ in the basis $E$ (thus the vector represents $a \cosh x + b \sinh x$) and returns the vector in the basis $F$.

\[
a \cosh x + b \sinh x = a \left( \frac{e^x + e^{-x}}{2} \right) + b \left( \frac{e^x - e^{-x}}{2} \right)
\]
\[
= \left( \frac{a + b}{2} \right) e^x + \left( \frac{a - b}{2} \right) e^{-x}
\]

Thus, we want

\[
T \left( \begin{array}{c} a \\ b \end{array} \right) = \left( \begin{array}{c}
\frac{1}{2} a + \frac{1}{2} b \\
\frac{1}{2} a - \frac{1}{2} b
\end{array} \right)
\]

Thus, $T = \left( \begin{array}{cc} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right)$

(b) To find the matrix representing the operator $D$, we apply the operator $D$ to each of the basis vectors in $E$, and make the result into a vector in basis $E$.

$[D(\cosh x)]_E = [\sinh x]_E = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$

$[D(\sinh x)]_E = [\cosh x]_E = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$

The resulting vectors are the columns of the matrix representing the operator $D$ with respect to the basis $E$. 

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Thus,

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

(c) We use the same procedure as in part (b):

\[ [D(e^x)]_F = [e^x]_F = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ [D(e^{-x})]_F = [-e^{-x}]_F = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \]

Thus,

\[ B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

4. Find the eigenvalues and eigenvectors of the following matrix.

\[ A = \begin{pmatrix} -3 & -1 & -3 \\ 0 & -1 & 0 \\ 6 & 1 & 6 \end{pmatrix} \]

We find the eigenvalues by solving for \( \lambda \) such that \( \det(\lambda I - A) = 0 \):

\[
\det(\lambda I - A) = \begin{vmatrix} \lambda + 3 & 1 & 3 \\ 0 & \lambda + 1 & 0 \\ -6 & -1 & \lambda - 6 \end{vmatrix} = (\lambda + 1) [((\lambda + 3)(\lambda - 6) + 18] = (\lambda + 1)(\lambda^2 - 3\lambda) = \lambda(\lambda + 1)(\lambda - 3)
\]

Thus, if \( \det(\lambda I - A) = 0 \), we have \( \lambda = 0, -1, \) or 3. These are the eigenvalues of \( A \).

For each of these eigenvalues, we compute an eigenvector:
• $\lambda = 0$

$$\text{nullspace} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 0 \\ -6 & -1 & -6 \end{pmatrix} = \text{nullspace} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Thus, an eigenvector corresponding to the eigenvalue $\lambda = 0$ is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Any multiple of this vector is also an eigenvector.

• $\lambda = -1$

$$\text{nullspace} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ -6 & -1 & -7 \end{pmatrix} = \text{nullspace} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, an eigenvector corresponding to the eigenvalue $\lambda = -1$ is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Any multiple of this vector is also an eigenvector.

• $\lambda = 3$

$$\text{nullspace} \begin{pmatrix} 6 & 1 & 3 \\ 0 & 4 & 0 \\ -6 & -1 & -3 \end{pmatrix} = \text{nullspace} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$
Thus, an eigenvector corresponding to the eigenvalue $\lambda = 3$ is
\[
\begin{pmatrix}
-1 \\
0 \\
2
\end{pmatrix}
\]. Any multiple of this vector is also an eigenvector.