

The birth of Combinalysis

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My dear friend, I have made some new discoveries in analysis [2, page 278]. Even more fascinating than the discoveries themselves, which truly are of great import, is the method which occurred to me as I engaged in speculations on the matter. I have derived much satisfaction from the whole endeavor and I cannot but share my thoughts with you. It has been made obvious to me that I am standing at the door of a new field that will bring momentous enrichment to our knowledge of philosophy.

You may recall, as I happened to share this with you before, in the recent years my students started asking me questions the nature of which and the interest they aroused in them I could not understand for a long time. Dear teacher, they would ask, how many ways are there to arrange n things, whatever they may be, in a row? How many collections of k things one can choose out of n given ones? I used to brush off such questions without too much thought, for what can possibly be interesting, important or difficult about counting? Alas my friend, year after year they kept asking the same questions. They have the ability, but they are still not good geometers (which is, as you know, a great defect) [1, page 552]. However, they prosecuted this matter with such conviction and vigor that I finally found myself pondering over the issue, fully expecting that I would demonstrate that such questions are not worthy of the time and magnitude of a true practitioner.

Indeed, how many ways are there to arrange n things, whatever they may be, in a row? The easiest way to speculate about this topic is to lay down a law that establishes a pairing between such arrangements and particular regions, which I will describe promptly, inside the unity cube $I^n = \{(x_1, x_2, \dots, x_n) | 0 \leq x_1, x_2, \dots, x_n \leq 1\}$ of n di-

mensions. By this law each arrangement $t_{i_1}, t_{i_2}, \dots, t_{i_n}$ of the n things is paired up with the region $R(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ that consist of the points (x_1, x_2, \dots, x_n) in the unity cube whose components respect the order

$$0 < x_{i_1} < x_{i_2} < \dots < x_{i_n} < 1.$$

Different arrangements are, of necessity, paired up with different regions, but nevertheless the different regions have the same volume. Without a doubt, the different regions can be transformed at will into each other through simple matrix operations that rename the components and such operations are known not to change the volume. For what's in a name [5, Scene II, Act II]?. For the sake of our friendship, I will spare you of the dullness of the following calculation, which I intend to use freely as needed in the further deliberations.

Principium 1.

$$\int_a^b \int_a^{x_n} \int_a^{x_{n-1}} \dots \int_a^{x_2} dx_1 \dots dx_{n-2} dx_{n-1} dx_n = \frac{(b-a)^n}{n!}.$$

To render certain our understanding, $n!$ denotes the product of all terms of the arithmetic progression which begins with n , that is the number of things to be arranged, and decreases by 1 until the number 1 is reached [3, page 230]. Therefore, I reach the irrefutable conclusion that the volume of each of the regions is just the same as the volume of $R(x_1, x_2, \dots, x_n)$, which is now seen to be $1/n!$, and, furthermore, since all these regions compose the whole volume of the unity cube, which is 1, there must be $n!$ regions.

Principium 2. *The number of arrangements of n things in a row, whatever they may be, is $n!$.*

My dear friend, you know how badly I used to depend on games of chance for entertainment and how much trouble and loss this habit had brought upon me. This time, the truth be told, it worked to my advantage, for I immediately asked an infinitely more complex question to challenge myself. I am sure that I can make it understood, but it requires few words from me and a little patience from you [1, page 555]. For the sake of

an argument, say that I have n things, whatever they may be, and I decide to arrange them by chance in r places, with chances proportional to the capacity of the places. If $n = n_1 + n_2 + \cdots + n_r$ is a partition of n into r parts, what are the chances that n_j of the things will end up in the j -th place, the capacity of which is c_j , and this is so for all places?

The chances of a single thing being put in the j -th place are $p_j = c_j/c$, where c is the capacity of all places put together, that is $c = c_1 + c_2 + \cdots + c_r$. Apropos, I found it helpful to write s_j for the sum of the first j chances, e.g., $s_3 = p_1 + p_2 + p_3$.

Choose a point $A = (a_1, a_2, \dots, a_n)$ by chance from the unity cube I^n of n dimensions. I lay down a rule now that pairs the points in the cube with the arrangements of the n things in the r places. Relegate the i -th thing in the j -th place whenever the i -th component of the chosen point A is smaller than the sum of the first j chances, but is greater than the sum of the first $j - 1$ chances, that is $s_{j-1} < a_i < s_j$. Your keen eye and sharp mind, my friend, will notice that not all points in the cube are accounted for, but you will also soon admit that the omitted points, having no volume, bear no ill effect to my reasoning. Thus the points in the cube tell ways of arranging the n things in the r places and, in particular, so do the points in the region $R(x_1, x_2, \dots, x_n)$ intimated before. For a moment let me be allowed to choose points solely from this region. Given that n_j things are needed in the j -th place, the favorable points (that would accomplish this!) come from the region that consists of the points (x_1, x_2, \dots, x_n) within the inequalities

$$\begin{aligned}
 0 < x_1 \leq x_2 \leq \cdots \leq x_{n_1} < s_1 \\
 s_1 < x_{n_1+1} \leq x_{n_1+2} \leq \cdots \leq x_{n_1+n_2} < s_2 \\
 &\dots \\
 s_{r-1} < x_{n_1+\cdots+n_{r-1}+1} \leq x_{n_1+\cdots+n_{r-1}+2} \leq \cdots \leq x_n < 1.
 \end{aligned}$$

The volume of this favorable region is described by

$$\int_{s_{r-1}}^1 \int_{s_{r-1}}^{x_n} \cdots \int_0^{s_1} \int_0^{x_{n_1}} \cdots \int_0^{x_2} dx_1 \cdots dx_{n_1-1} dx_{n_1} \cdots dx_{n-1} dx_n,$$

and is easily found to be

$$\frac{p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}}{n_1! n_2! \cdots n_r!}.$$

The matrix operations that rename the components once again come to my rescue and assure you that the volume of the favorable regions throughout the whole cube I^n is then just $n!$ times greater. The numbers

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

appear often enough in my considerations to deserve their own symbol $\binom{n}{n_1, n_2, \dots, n_r}$ and to be named multinomial coefficients (the reason for the name will soon be revealed). Thus I established the following truth, applicable to cases innumerable.

Principium 3. *Let a game of chance have r possible outcomes, with chances p_1, p_2, \dots, p_r . If the number of games to be played is called n and $n = n_1 + n_2 + \cdots + n_r$, then the chances of producing the j -th outcome exactly n_j times, and this being so for all outcomes, are*

$$\binom{n}{n_1, n_2, \dots, n_r} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}.$$

Deeper and deeper truths started now multiplying and jumping around much as effortlessly as rabbits in springtime. For example, I have conceived then of a wonderful rule for the coefficients of powers not only of the binomial $x + y$, but also of the trinomial $x + y + z$, in fact, of any polynomial; so that when given the power of any degree say the tenth, and any term contained in it, as $x^5 y^3 z^2$, it should be possible to assign the coefficient (numerus coefficientem) which it must have [3, page 229]. You allow me to be concise to make myself understood to a man who comprehends the whole from half a word [1, page 563]. Given that the chances must accumulate to 1, the following principium is now self-evident.

Principium 4. *Let c_1, c_2, \dots, c_r be non-negative, but otherwise arbitrary, capacities. Then, for all natural powers n ,*

$$(c_1 + c_2 + \cdots + c_r)^n = \sum_{\text{all partitions of } n \text{ in } r \text{ parts}} \binom{n}{n_1, n_2, \dots, n_r} c_1^{n_1} c_2^{n_2} \cdots c_r^{n_r}.$$

When a number is partitioned in parts I allow each part to be measured by a natural number (as we always agreed, natural numbers include the empty number, despite of the fact that this proposition is still not resolved to a satisfactory degree).

The above arguments explain sufficiently the name multinomial coefficients, but before I can reveal to you the true meaning of such quantities I put forward this triviality.

Principium 5. *There are r^n arrangements of n things in r places, the things being distinguishable, as well as the places.*

You will easily see that I have not committed a paralogism here [1, page 556]. The fact is, I have already told you how to pair up points in the cube I^n and arrangements of the things in the places. The rule was in concordance with the capacities of the places, which I now assume are the same for all places. Different arrangements, by necessity, correspond to different regions, but all regions are copies, of translated nature, of a cube of size $1/r$ with volume

$$\int_0^{1/r} \int_0^{1/r} \dots \int_0^{1/r} dx_1 \dots dx_{n-1} dx_n = \frac{1}{r^n}.$$

These regions compose together the whole unity cube I^n , so there must be r^n of them.

Having learned this, the other could not long remain hidden from me [4, page 227]. I shall answer now the question of my students that asked how many collections of k things one can choose from a collection of n given ones; in fact, I shall answer more than what was asked.

Principium 6. *Let us agree that $n = n_1 + n_2 + \dots + n_r$. The number of arrangements of n things in r places so that the j -th place contains n_j things, and this is so for all places, is the multinomial coefficient*

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}.$$

The last principium is in consequence of the truth that there are r^n arrangements and, furthermore, the chances of choosing one that corresponds to the partition $n = n_1 + n_2 + \dots + n_r$ are $\binom{n}{n_1, n_2, \dots, n_r} \frac{1}{r^n}$ when all places have the same capacity.

Further truths can now be deducted at pleasure, the nature of which is combinatorial, but the methods of deduction come from analysis, and this is why I called this new field combinalysis. I readily admit that I found this new subject extraordinary promising and I expect many more and much deeper truths to be discovered as a result of further speculations. But I have no time, and my ideas are not developed in this field, which is immense [2, page 285].

But enough of geometry, as much as it is dear to my heart, I am afraid lest I have some less pleasant things to share.

Comments from our historian

The letter is either not finished or the last part is lost. Despite of a lot of effort, the author of the letter could not be identified or placed well in time and space. Nevertheless, the research showed that whole sentences from the letter were reproduced, almost word for word in letters by various mathematicians over the centuries, who were obviously aware of the existence of the original letter and were greatly influenced by it. Each such sentence is indicated above by a corresponding reference at the end of the sentence. It is noteworthy, if not entirely perplexing, that both Fermat and Pascal as well as both Leibniz and Jean Bernoulli used parts from the original letter in their mutual correspondence. It is also unfathomable why these renowned mathematicians used the results given in the letter and some sentence constructions, while the peculiar, but powerful, geometric and probabilistic methods used by the letter's author were inexplicably ignored.

A yet another theory goes to the other extreme to claim that the letter is just another quasi-postmodern (if such a thing is possible) attempt to blur the issues by tweaking the history (but this does not explain well the originality of the given proofs and is a rather cynical point).

References

- [1] Pierre de Fermat and Blaise Pascal, correspondence, 1654, cited from D. E. Smith, *On probability*, A source book in mathematics, Dover Publications, New York, 1959, 546–565.
- [2] Évariste Galois, letter to Auguste Chevalier, 1832, cited from D. E. Smith, *On groups and equations and Abelian integrals*, A source book in mathematics, Dover Publications, New York, 1959, 278–285.
- [3] Gottfried Wilhelm Freiherr von Leibniz and Jean Bernoulli, correspondence, 1695, cited from D. E. Smith, *On the polynomial theorem*, A source book in mathematics, Dover Publications, New York, 1959, 229–230.
- [4] Isaac Newton, letters to Oldenburg to be transmitted to Leibniz, 1676, cited from D. E. Smith, *On the binomial theorem for fractional and negative exponents*, A source book in mathematics, Dover Publications, New York, 1959, 224–228.
- [5] William Shakespeare, *Romeo and Juliet*, retrieved on April 12, 2002 from http://the-tech.mit.edu/Shakespeare/romeo_juliet/

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