

MATRIX INVERSION AND THE GREAT INJUSTICE

ZORAN ŠUNIĆ

Dear Editor, I am writing this letter to you in the hope that you or your readers can help me with this matter. A great injustice has been done and I would appreciate any help I can get. I desperately need to get an A in this course in order to bump up my GPA enough to get accepted in my major. Unfortunately, the teacher says he could not give me anything better than A-. But, let me explain first what happened.

A couple of days ago, the following problem was given to us on the test.

Problem Determine the inverse matrix of

$$M = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}.$$

Now, this was very unfair question. You see, we all expected to be given a 3×3 matrix. I practiced and practiced on the 3×3 case and I remembered everything so well. You cross the rows, you cross the columns, calculate the determinants of the remaining parts, switch the signs according to the checkerboard pattern, you flip the whole thing and you are done. When he was teaching, the teacher used some fancy words like minors, factors, adjuncts and whatnot, but believe me, what I said before is all you have to do to get the inverse right. Oh yes, divide by the determinant at the end.

If you are confused by my explanations you can check what my textbook says about this method [1, Section 3.3].

I was sitting there and I tried to remember how to do the 2×2 case in a similar fashion. However, if I cross a row and a column all that is left is one number and I certainly cannot calculate a determinant when there is no matrix.

So I tried another thing. I remembered that the inverse matrix must be a matrix

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

of size 2×2 such that

$$MX = I,$$

where I is the indennity matrix. When I multiplied the left hand side I got

$$\begin{bmatrix} x_1 + 3x_3 & x_2 + 3x_4 \\ x_1 + x_3 & x_2 + x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and when I set the corresponding entries equal I obtained the following system

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array} \right].$$

At this point I again realized that I was in trouble. I knew there was this messy method of solving any system by somehow making some entries 1 and some 0, but

it was too complicated and I never quite understood which entries had to be made 1 and which 0. The teacher was probably unsure too, because he made some entries 1 in some problems and made them 0 in others. This is supposedly explained in the very first chapter of my textbook [1, Section 1.2] and my roommate claims that she actually got it from there. But she claims so many things.

Then I remembered Cramer's Rule. I named the 4×4 matrix of the system by A and I knew that I needed to calculate $\det(A)$. Oh, no! How do I calculate a 4×4 determinant? The teacher said we were not going to be given 4×4 determinants on this test, we only had to know the 3×3 and 2×2 cases. Then I said to myself, why not use the same method that works for 3×3 ? I rewrote the first three columns of A to the right of A , calculated the four products going down and to the right with plus sign

$$\begin{array}{cccccccc}
 & + & & + & & + & & + & & \\
 & 1 & & 0 & & 3 & & 0 & & 1 & & 0 & & 3 \\
 & & \backslash & & \backslash & & \backslash & & \backslash & & \backslash & & \backslash & \\
 & & 0 & & 1 & & 0 & & 3 & & 0 & & 1 & & 0 \\
 & & & & & \backslash & & \backslash & & \backslash & & \backslash & & \backslash & \\
 & & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 \\
 & & & & & & & \backslash & & \backslash & & \backslash & & \backslash & \\
 & & & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0
 \end{array}$$

and the four products going down and to the left with minus sign

$$\begin{array}{cccccccc}
 & & & & & - & & - & & - & & - & & - & \\
 & 1 & & 0 & & 3 & & 0 & & 1 & & 0 & & 3 & & \\
 & & & & & & / & & / & & / & & / & & / & \\
 & & & 0 & & 1 & & 0 & & 3 & & 0 & & 1 & & 0 \\
 & & & & & & & / & & / & & / & & / & \\
 & & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 \\
 & & / & & / & & / & & / & & / & & / & \\
 & & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0
 \end{array}$$

and I obtained $\det(A) = 1 + 0 + 9 + 0 - 3 - 0 - 3 - 0 = 4$. In exactly the same fashion I calculated the other four determinants that I needed for the Cramer's Rule, namely the determinants $\det(A_1) = -2$, $\det(A_2) = 6$, $\det(A_3) = 2$ and $\det(A_4) = -2$. Then I calculated

$$\begin{aligned}
 x_1 &= \frac{\det(A_1)}{\det(A)} = -\frac{1}{2} & x_2 &= \frac{\det(A_2)}{\det(A)} = \frac{3}{2} \\
 x_3 &= \frac{\det(A_3)}{\det(A)} = \frac{1}{2} & x_4 &= \frac{\det(A_4)}{\det(A)} = -\frac{1}{2}
 \end{aligned}$$

and concluded that

$$M^{-1} = X = \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

I was very tired by the end of these inextricable calculations, but I managed to convince myself to check the result at the end, mainly because I heard that the teacher usually gave partial credit for almost anything the students wrote down. And there it was! It checked out nicely, I actually solved the problem, I was actually good in math after all. I knew that all the time, but I had very bad teachers before and it took me a while to realize that ... Oh, where was I?

Anyway, let me go back to my case. The teacher gave me 7 out of 10 points for my solution. I complained and complained, of course, I had correct solution, I explained and explained to my teacher that everything I did was good, but he did not want to listen. He said that, for the most part, he was very happy with my work, but I still did not deserve the full credit. For example, he did not like the fact that I did not know the simple way to do the problem, but he liked that, rather than remembering formulae, I actually knew what an inverse matrix was. Introducing unknowns and setting a system to find them was a lovely idea, he said. I should have used the row eschalot matrix to solve the system though, he said, and he was not a big fan of Cramer's Rule either. Nevertheless, he was delighted that I used it properly, because he never mentioned that it works for more than three unknowns. Then, he shook his head and said that it was unfortunate that I did not know how to calculate determinants by expansion, and I still have no clue what he meant by that. He concluded by trying to convince me that the method I used to calculate determinants was not good at all. I was just lucky to get the correct answer, he said. That trick with plus products going to the right and minus products going to the left worked only for matrices of order 3 (it took me a while to realize that that's his fancy way of calling the 3×3 matrices). He even showed me an example of a 4×4 matrix for which he claimed that the trick did not work. He had me right there, what could I say? I did not know how to calculate determinants by his expansions and I had to believe him.

I went home, watched TV with my roommate and tried to forget the whole thing. My roommate is a very annoying person, and she kept asking me about the exam. She is a math major and she actually enjoys talking math so she insisted and insisted, until I told her about it. I did not have the test with me, I threw it as soon as I left the teachers office, of course, and I just put some random numbers for the entries in the matrix M , just to explain what I did. Sure enough, all these math people are the same, as soon as I explained how I calculated the 4×4 determinants she screamed "But no, you cannot do that." When I asked why not, she said "Finish the problem just like you did in your test and check the matrix at the end. You were lucky with the matrix on the test, but since this is a different matrix it is very unlikely that you will be lucky again." I calculated everything for this new matrix and checked the answer. And there it was! It worked! She did not believe me and she checked everything, but she got the same result. Then she tried different and different examples and my method always gave the correct answer. Always! For all examples we tried.

My roommate tried to discourage me yesterday by showing me an example of a 3×3 matrix M for which my method of matrix inversion failed. She introduced a matrix X with nine unknowns, got a 9×9 system, calculated the needed determinants by extending the matrix of the system by its first 8 columns and then using the 9 products going down and to the right with plus sign and the 9 products going down and to the left with minus sign, and finally she used the Cramer's Rule to get the entries of X , but the final answer did not check out.

Despite all this, she never found an example of a 2×2 matrix for which my method failed. So I am asking you, does my method always give the correct inverse matrix in the 2×2 case? And if it does, then it is a correct method, isn't it? Why did the teacher take off my three points then? Let me remind you that all this is not just an academic discussion. Without the three points I am out of the A range

in the course and I cannot get into my major. This ticks me off so much and I became so interested and involved that all I could do for the last three days was think about it and try to prove that my method works.

What do you think?

REFERENCES

1. David Lay, *Linear Algebra and its Applications*, second edition, Addison Wesley, 1996.

Answer from the Editor: Dear Zoran, your method presents a startling, idiosyncratic, convoluted and perfectly valid way to invert a matrix of order 2 (I prefer the fancy way too). Still, your teacher treated and graded you fairly. Rather than explaining how can you both be right let me move on to the important thing. Forget about the three points. Your letter shows some of your weaknesses (indemnity matrix? eschalot form? inability to perform row reduction! etc.), but it also shows signs of understanding, persistence, curiosity and brilliance that would lead you to a very successful career in mathematics. Go for it!

DEPARTMENT OF MATHEMATICAL SCIENCES, 810 OLDFATHER HALL, UNIVERSITY OF NEBRASKA,
LINCOLN, NE 68588-0323, ZSUNIK@MATH.UNL.EDU