A PASTICHE ON EMBEDDINGS INTO SIMPLE GROUPS
(FOLLOWING P. E. SCHUPP)

ZORAN ŠUNIĆ

Abstract. Let $\lambda$ be an infinite cardinal number and let $C = \{H_i \mid i \in I\}$ be a family of nontrivial groups. Assume that $|I| \leq \lambda$, $|H_i| \leq \lambda$, for $i \in I$, and at least one member of $C$ achieves the cardinality $\lambda$.

We show that there exists a simple group $S$ of cardinality $\lambda$ that contains an isomorphic copy of each member of $C$ and, for all $H_i, H_j$ in $C$ with $|H_i| = \lambda$, is generated by the copies of $H_i$ and $H_j$ in $S$.

This generalizes a result of Paul E. Schupp (moreover, our proof follows the same approach based on small cancelation). In the countable case, we partially recover a much deeper embedding result of Alexander Yu. Ol’shanskiĭ.

1. Background and results

In [Sch76] Schupp used small cancelation theory (construction of Adian-Rabin type) to prove, among other things, the following result.

Theorem S (Schupp [Sch76]). Let $G, H$ and $K$ be nontrivial groups with $|G| \leq |H \ast K|$ and $|K| \geq 3$. There exists a simple group $S$ that contains an isomorphic copy of $G$ and is generated by isomorphic copies of $H$ and $K$.

Corollary S. Let $G$ be a countable group. For all $p, q \in \{2, 3, \ldots\} \cup \{\infty\}$ with $q \geq 3$, there exists a simple group $S$ that contains an isomorphic copy of $G$ and is generated by a pair of elements whose orders are $p$ and $q$, respectively.

The simple group constructed by Schupp in Theorem S, in addition to being dependent on $G$, depends on $H$ and $K$. Accordingly, the simple group in Corollary S, in addition to being dependent on $G$, depends on the pair $(p, q)$.

We will show that the argument used by Schupp can be adapted in such a way that the same simple group can be used even if one considerably varies $H$ and $K$ in Theorem S and, consequently, the same simple group can be used independently of the pair $(p, q)$ in Corollary S.

Theorem A. Let $|I| \geq 2$ and $C = \{H_i \mid i \in I\}$ be a countable family of countable nontrivial groups, at least one of which has at least 3 elements (the groups may be isomorphic for different values of the index).

There exists a 2-generated simple group $S$ that contains an isomorphic copy of each member of $C$ and, for all $H_i, H_j$ in $C$ with $|H_j| \geq 3$, is generated by the copies of $H_i$ and $H_j$ in $S$.

2000 Mathematics Subject Classification. 20F06, 20E32.

Key words and phrases. simple groups, embeddings, small cancelation.

Partially supported by NSF grant DMS-0600975.
Corollary A. Let \( G \) be a countable group.

There exists a simple group \( S \) that contains an isomorphic copy of \( G \) and, for all \( p, q \in \{2, 3, \ldots \} \cup \{\infty\} \) with \( q \geq 3 \), is generated by a pair of elements whose orders are \( p \) and \( q \), respectively.

Moreover, if \( |G| \geq 3 \), then, for every \( p \in \{2, 3, \ldots \} \cup \{\infty\} \), the simple group \( S \) is generated by \( G \) and an element of order \( p \).

In this note, countable means finite or countably infinite. The countability limitations imposed in Theorem A are natural since every countable group contains only countably many finitely generated subgroups. An extension of Theorem S in which countability assumptions are not used follows.

Theorem B. Let \( \lambda \) be an infinite cardinal number and let \( \mathcal{C} = \{H_i \mid i \in I\} \) be a family of nontrivial groups. Assume that \( |I| \leq \lambda \), \( |H_i| \leq \lambda \), for \( i \in I \), and at least one member of \( \mathcal{C} \) achieves the cardinality \( \lambda \).

There exists a simple group \( S \) of cardinality \( \lambda \) that contains an isomorphic copy of each member of \( \mathcal{C} \) and, for all \( H_i, H_{i'} \) in \( \mathcal{C} \) with \( |H_{i'}| = \lambda \), is generated by the copies of \( H_i \) and \( H_{i'} \) in \( S \).

Corollary B. For any group \( G \) with \( |G| \geq 3 \), there exists a simple group \( S \) that contains an isomorphic copy of \( G \) and, for every \( p \in \{2, 3, \ldots \} \cup \{\infty\} \), is generated by \( G \) and a single element of order \( p \).

In the countable case, the embedding results of Schupp were eventually subsumed by the following result of Ol’shanskiǐ (this result also subsumes our Theorem A, but not Theorem B).

Theorem O (Ol’shanskiǐ [Ol’89]). Let \( |I| \geq 2 \) and \( \mathcal{C} = \{H_i \mid i \in I\} \) be a countable family of countable nontrivial groups.

There exists a 2-generated simple group \( S \) that contains an isomorphic copy of each member of \( \mathcal{C} \) and, moreover, has the following properties (the copy of \( H_i \) in \( S \) is denoted by \( H_i \) below).

1. If \( i, j \in I \), \( i \neq j \), \( |H_j| \geq 3 \), then \( S \) is generated by \( H_i \) and \( H_j \).
2. If \( i, j \in I \), \( i \neq j \), then \( H_i \cap H_j = 1 \).
3. Every element of finite order in \( S \) is conjugate to an element in \( H_i \), for some \( i \in I \).
4. Every proper subgroup of \( S \) is either infinite cyclic, or infinite dihedral, or it conjugate of a subgroup of \( H_i \), for some \( i \in I \).
5. If, for some \( i \in I \), \( x \in H_i \), \( x \neq 1 \), \( y \notin H_i \), then either \( S \) is generated by \( \{x, y\} \) or both \( x \) and \( y \) are involutions, or both \( x \) and \( xy \) are involutions.
6. If \( i, j \in I \), \( i \neq j \), then \( H_i \cap H_j^x = 1 \), for every element of \( S \).
7. For every \( i \in I \), \( H_i \) is malnormal in \( S \) (for every \( x \in S \setminus H_i \), \( H_i \cap H_i^x = 1 \)).

Thus there is a natural trade off in our approach. We extend Theorem S of Schupp (by adapting his approach using small cancellation theory) to arbitrary families of groups in a way that, in the countable case, partially recovers Theorem O of Ol’shanskiǐ. A modest gain is achieved by the fact that the taken approach allows us to handle families of groups that are not necessarily countable. On the other hand, in the countable case, we recover only a small subset of the conclusions that are obtained by the more powerful (but also more onerous) graded diagram methods introduced by Olshanskiǐ.
2. Proofs and additional comments

Proof of Theorem A. Reindex the family $C$ (if necessary) so that it is indexed by an initial segment $I$ of the set of natural numbers $\mathbb{N} = \{0, 1, 2 \ldots \}$ (including the possibility $I = \mathbb{N}$, if $I$ is infinite). Moreover, in case the cyclic group $C_2$ of order 2 is a member of $C$ set $H_0 = C_2$ and make sure that this is the only copy of $C_2$ in $C$.

For each $i \in I$, embed $H_i$ into a 2-generated simple group $S_i = \langle s_i, t_i \rangle$ (this can be done by Theorem S) and consider the free product $F = A * B * \{s_i \in I S_i\}$, where $A = \langle a \mid a^2 \rangle = C_2$, $B = \langle b \mid b^3 \rangle = C_3$.

For each index $i \in I$ define the words

$$u_i = s_i (ab)^{(2i+1)n+1} (ab^{-1}) (ab)^{(2i+1)n+2} (ab^{-1}) \ldots (ab^{-1}) (ab)^{(2i+1)n+n},$$

$$v_i = t_i (ab)^{(2i+2)n+1} (ab^{-1}) (ab)^{(2i+2)n+2} (ab^{-1}) \ldots (ab^{-1}) (ab)^{(2i+2)n+n},$$

where $n$ is a positive integer to be specified at a later stage.

Choose a nontrivial element $h_0$ in $H_0$ and, for each $i > 0$, choose a pair of distinct nontrivial elements $h_i$ and $\bar{h}_i$ in $H_i$. For each pair of indices $i, j \in I$ with $0 \leq i < j$, define the words

$$w_{(a,i,j)} = a (h_i h_j) (h_i h_j)^2 (h_i h_j) \ldots (h_i h_j) (h_i h_j)^n,$$

$$w_{(b,i,j)} = b (h_i h_j)^{n+1} (h_i h_j) (h_i h_j)^{n+2} (h_i h_j) \ldots (h_i h_j) (h_i h_j)^{2n}.$$

Let $R$ be the set of words obtained by closure under inversion and cyclic conjugates of

$$R' = \{ w_{(a,i,j)}, w_{(b,i,j)} \mid i, j \in I, 0 \leq i < j \} \cup \{ u_i, v_i, (h_i a)^n, (h_i b)^n \mid i \in I \}$$

and let $H = ( F \mid R')$.

Choose $n$ that is relatively prime to 6 and is sufficiently large to ensure that the set of words $R$ satisfies the small cancelation condition $C'(1/6)$. It follows, by a result of Lyndon [Lyn66, Theorem IV] (see [LS01, Section V.9] for an exposition), that all factors in the free product $F$ are embedded in $H = ( F \mid R')$.

The $u$ relators and the $v$ relators ensure that $H$ is generated by $a$ and $b$. On the other hand, the $w$ relators ensure that $H$ is generated by $H_i$ and $H_j$ for any $i, j \in I$ with $0 \leq i < j$.

Let $M$ be a maximal normal subgroup of $H$ and let $S = H/M$. The group $S$ is simple by the maximality of $M$. We claim that all the factors $S_i$, $i \in I$, are still embedded in $S$. The factor $S_i$, being simple, either intersects $M$ trivially or is contained in $M$. In the former case, the factor $S_i$ is still embedded in $S = H/M$.

The latter case implies that $h_i = 1$ in $S$. Because of the relators $(h_i a)^n$ and $(h_i b)^n$, it follows that $a^n = b^n = 1$ in $S$. However, $n$ is chosen to be relatively prime to 6. Thus $a = b = 1$ in $S$, which means that $S$ is trivial, a contradiction.

This completes the proof. □

We note here the crucial role of the embeddings $H_i \hookrightarrow S_i$ in the course of the proof. On one hand, the number of generators needed for each factor is uniformized. This is notationally convenient, but not crucial. More significant is the simplicity of the factors $S_i$, which, together with the relators $(h_i a)^n$ and $(h_i b)^n$, “protect” the embedded subgroups $H_i$ from “crashing” when $M$ is factored out from $H$.

Proof of Corollary A. Apply Theorem A to $C = \{H_i \mid i \geq 1\}$, where $H_0 = C_2$, $H_1 = G$, and $H_{2i-4} = H_{2i-3} = C_i$, for $i \geq 3$. □
Proof of Theorem B. Let $J$ be an indexing set of cardinality $\lambda$. For each $i \in I$, embed $H_i$ into a simple group $S_i = \langle \{s_{i,j} \mid j \in J\} \rangle$. The cardinality of the generating system of $S_i$ can be chosen to be equal to $\lambda = |J|$ by Theorem S (we may deliberately choose redundant generators just to make the cardinality large enough; this is merely a notational convenience). Consider the free product $F = A \ast B = \langle \{a \mid a^2\} \rangle$, $B = \langle \{b \mid b^3\} \rangle = C_3$.

Let $\alpha : I \times J \to B$ be an injective map (such a map exists since $|I| \leq |J|$ and $|J|$ is infinite).

For each pair $(i, j) \in I \times J$, define the word

$$w_{i,j} = s_{i,j} (a_{\alpha(i,j)}) (a_{\alpha(i,j)}^{-1}) (a_{\alpha(i,j)^2}) (a_{\alpha(i,j)^3}) \ldots (a_{\alpha(i,j)^n}) (a_{\alpha(i,j)})^n,$$

where $n$ is a positive integer to be specified at a later stage.

For each $i \in I$, choose a nontrivial element $h_i$ in $H_i$. For each $i' \in I$ such that $|H_i'| = \lambda$, choose a nontrivial element $h_{i'}$ in $H_{i'}$ different from $h_i$ and distinct nontrivial elements $h_{i', j}$, $j \in J$, $h_{i', j}$, $j \in J$, in $H_i'$ that are also different from $h_i$ and $h_{i'}$. Let $L \subseteq I \times I$ be a set of pairs such that, for each pair of indices $i, i' \in I$ such that $|H_i| < |H_i'| = \lambda$, the ordered pair $(i, i')$ is in $L$, and, for each pair of indices $i, i' \in I$ such that $i \neq i'$ and $|H_i| = |H_i'| = \lambda$, exactly one of the ordered pairs $(i, i')$ and $(i', i)$ is in $L$. For every pair $(i, i')$ in $L$, define the words

$$w_{(a,i,i')} = a (h_i h_{i'}) (h_i h_{i'})^2 (h_i h_{i'}) \ldots (h_i h_{i'}) (h_i h_{i'})^n,$$

$$w_{(b,i,i')} = b_j (h_i h_{i'}) (h_i h_{i'}) (h_i h_{i'})^2 (h_i h_{i'}) \ldots (h_i h_{i'}) (h_i h_{i'})^n, \quad j \in J.$$

For all $i \in I$ and $j \in J$ also define the words

$$(h_i a)^n, \quad (h_i b_j)^n.$$

The group $H = F/R$ is defined as before ($R$ is obtained by symmetrizing all the words defined so far and $n$ is chosen to be relatively prime to 6 and to be sufficiently large to yield the cancelation condition $C'(1/6)$). The group $S = H/M$, where $M$ is a maximal normal subgroup of $H$ satisfies the required conditions.

\[\Box\]

Remark 1. We did not specify a value for $n$ in the proof of Theorem A. We quickly indicate here that $n = 23$ is a good choice.

The lengths of the words in the set of relators $R'$ are

$$|u_i| = 4n^2i + 3n^2 + 3n - 1, \quad |w_{(a,i,j)}| = n^2 + 3n - 1, \quad |(h_i a)^n| = 2n,$$

$$|v_i| = 4n^2i + 5n^2 + 3n - 1, \quad |w_{(b,i,j)}| = 3n^2 + 3n - 1, \quad |(h_i b_j)^n| = 2n.$$

It is easy to check that the longest piece of $w_{(a,i,j)}$ is the word $(h_i h_j)^{n-1}(h_i h_j)_{n}^n$, whose length is $4n$. Thus $n$ needs to be selected in such a way that

$$\frac{4n}{n^2 + 3n - 1} < \frac{1}{6}.$$

This is true for any $n \geq 22$, but since we require $n$ to be relatively prime to 6, the smallest good choice is $n = 23$.

One can now fix $n = 23$, consider all other pieces of words, and check that the $C'(1/6)$ condition is satisfied.

For instance, the word $(ab)^{(2n+1)n+n-1}(ab^{-1})(ab)^{(2n+1)n+n}$ of length $8ni + 8n$ is the longest piece of $u_i$. Thus we need to check that

$$\frac{8ni + 8n}{4n^2i + 3n^2 + 3n - 1} < \frac{1}{6}.$$
for all $i \geq 0$. Think of the fraction on the left as a function of $i$. Since $8n(3n^2 + 3n - 1) - 8n\cdot 4n^2 < 0$ the maximum is achieved at $i = 0$ and its value is $8n/(3n^2 + 3n - 1) = 184/1655 < 1/6$.

We can equally easily check all other cases.

**Remark 2.** Consider again the proof of Theorem A. We used the original work of Schupp not only to model our approach, but also to embed each group $H_i$ into a simple $2$-generated group $S_i$ (in order to protect $H_i$ in the quotient $S = H/M$). In turn, in his proof of Theorem $S$, Schupp uses embeddings of $G$, $H$, and $K$ into countable simple groups. At about the same time Schupp proved his result, Goryushkin also proved that every countable group can be embedded into a $2$-generated simple group [Gor74]. Before the results of Schupp and Goryushkin, it was known from the work of P. Hall that every countable group can be embedded into a $3$-generated simple group [Hal74, Theorem C2]. However, both Hall and Goryushkin also base their proofs on the existence of embeddings of countable groups into countable simple groups. Thus to get back on some firm footing one could perhaps directly go back to the classical embedding results of Higman, Neumann and Neumann. Namely they prove [HNN49] that every countable group can be embedded into a countable group in which any two elements that have the same order are conjugate. As a corollary, every countable group can be embedded into a countable simple divisible group (see [LS01, Theorem IV.3.4] for an exposition). Of course, using such embeddings directly in the course of the proof of Theorem A would skip over a layer in the construction at the cost of a mild notational difficulty (one would have to deal with countably many countable generating sets).

**References**


Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

E-mail address: sunic@math.tamu.edu