The "implicit differentiation problem" below is slightly too complex to discuss it in lecture, but you must be able to follow the arguments. A less genereal point than the one in Exercise 25 in the book has been picked.

Let $C := \{(x,y) : 2(x^2+y^2)^2 = 25(x^2-y^2)\}$. What are the equations for the tangent lines to C at $(0,0)$?

Solution. As the formula

$$y := \pm\sqrt{\frac{-(4x^2+25)+\sqrt{(4x^2+25)^2-8(2x^4-25x^2)}}{4}}$$
$$= \pm x \left(\frac{8(25-2x^2)}{4\left((4x^2+25)+\sqrt{(4x^2+25)^2-8(2x^4-25x^2)}\right)} \right)^{1/2}$$

shows, $y'(0)$ exists, $y(0)=0$, and $y(x)$ is continuous at 0 (why?). We have

$$y'(0) = \lim_{x\to 0} \frac{y-0}{x-0} = \lim_{x\to 0} \frac{y}{x}\,.$$

Dividing both sides of the equation defining C by x^2, we obtain

$$2(x^2+y^2)\left(1+\left(\frac{y}{x}\right)^2\right) = 25\left(1-\left(\frac{y}{x}\right)^2\right)\,.$$

Taking the limit when $x \to 0$ on both sides yields

$$2\lim_{x\to 0}(x^2+y^2)\lim_{x\to 0}\left(1+\left(\frac{y}{x}\right)^2\right) = 25\lim_{x\to 0}\left(1-\left(\frac{y}{x}\right)^2\right).$$

Hence

$$0\left(1+\left(\lim_{x\to 0}\frac{y}{x}\right)^2\right) = 25\left(1-\left(\lim_{x\to 0}\frac{y}{x}\right)^2\right)\,.$$

Note that the left hand side is 0 since the limit in the second factor is finite (it is $y'(0)$). Our concluding statement is

$$y'(0) = \lim_{x\to 0}\frac{y}{x} = \pm 1\,,$$

and the equations for the tangent lines to C at $(0,0)$ are $y = \pm x$.