

Newton Polytopes via Numerical Algebraic Geometry

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(Work with Jonathan Hauenstein and Frank Sottile)

AMS sectional at University of Denver

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Given a hypersurface, get information about its defining polynomial.

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This talk: Algorithm by Hauenstein and Sottile [3]
Implementation in Macaulay2 using Bertini.m2 [1],[2].

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$$f(x, y) = 4y + 8x^2 + 15x^3 + 16x^2y^2 + 23x^4y + 42x^2y^4$$



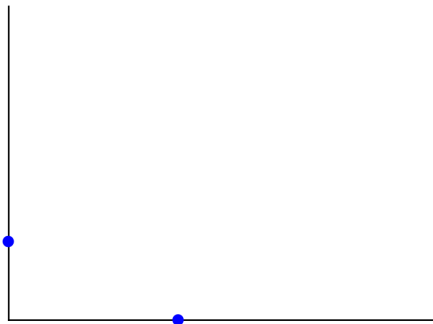
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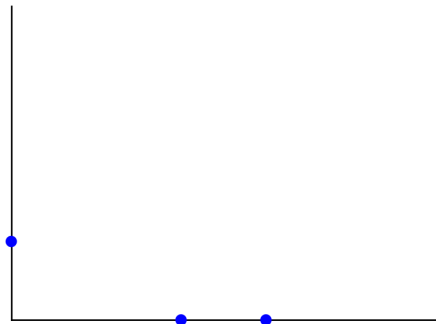
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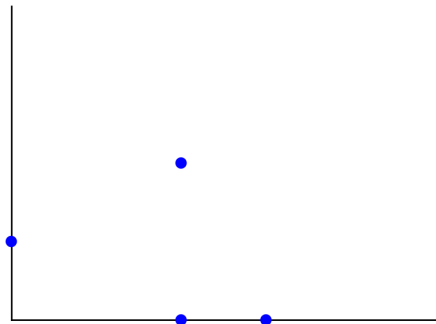
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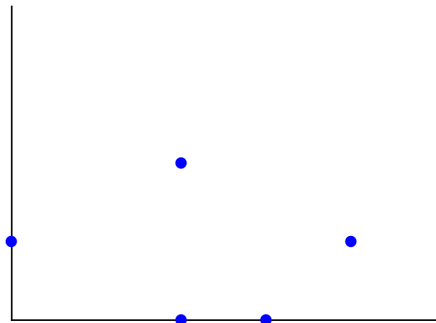
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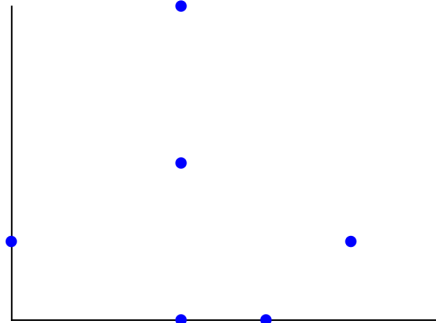
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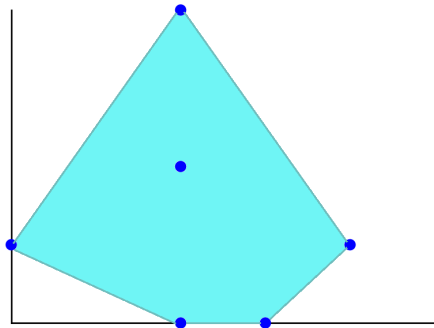
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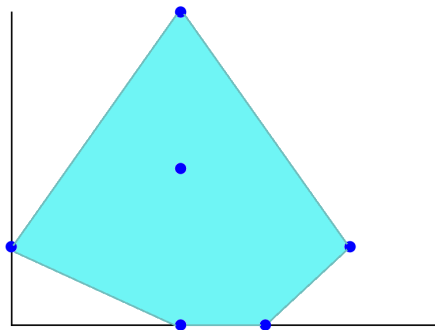
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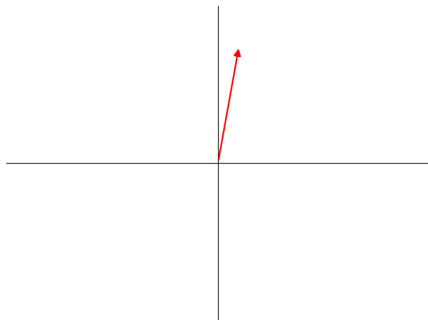
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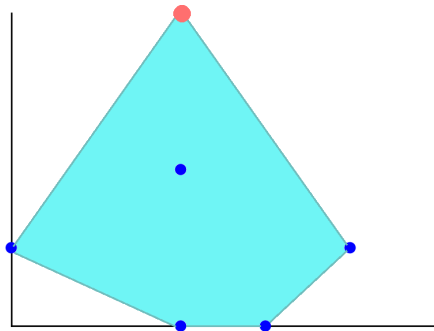


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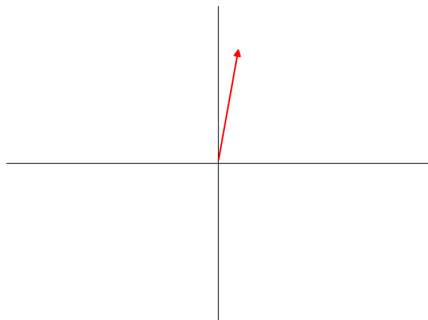
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$$f_{(1,6)}(x, y) = 42x^2y^4$$

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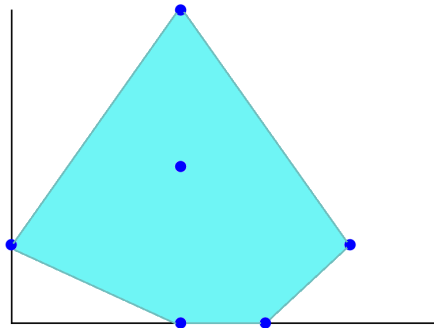
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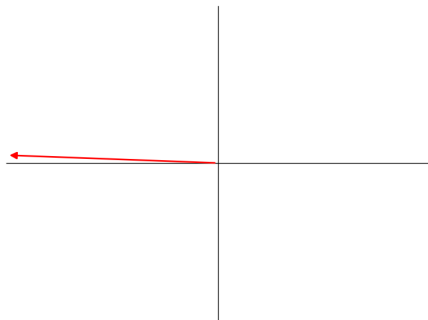
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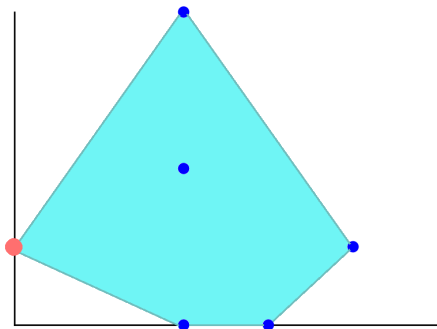


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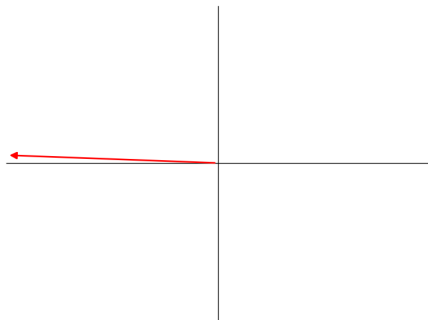
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$$f_{(-7,1)}(x, y) = 4y$$

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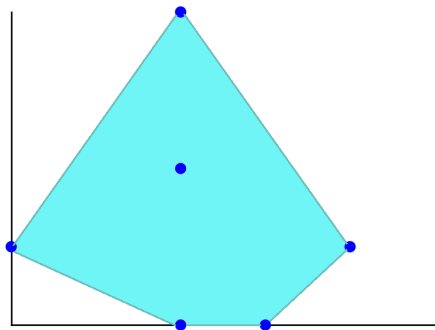
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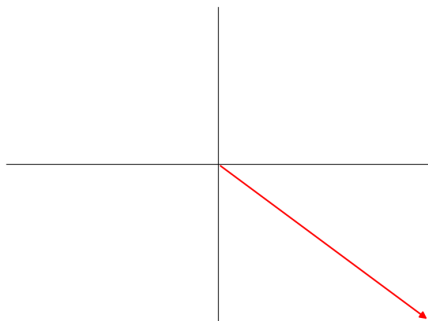
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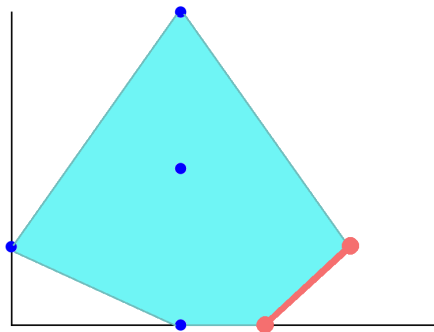


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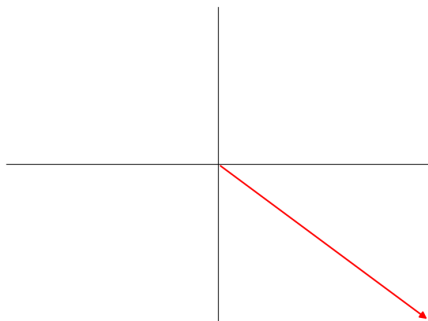
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$$f_{(1,-1)}(x, y) = 15x^3 + 23x^4y$$

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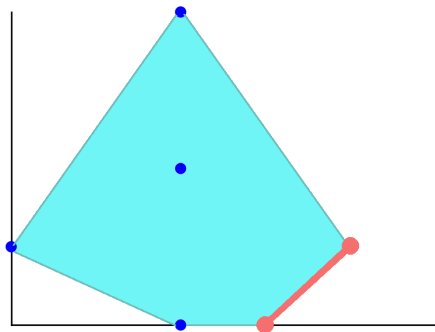
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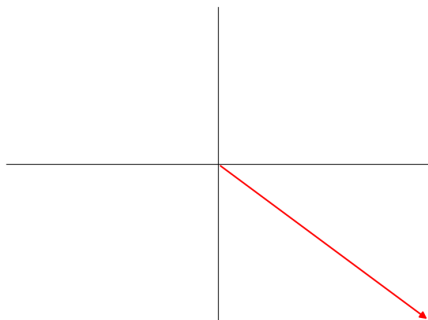
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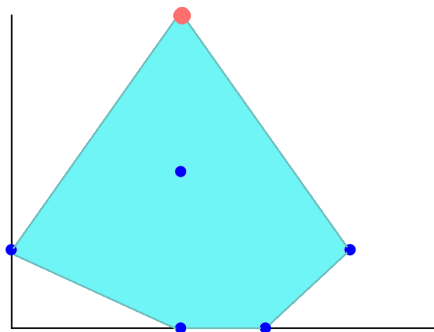


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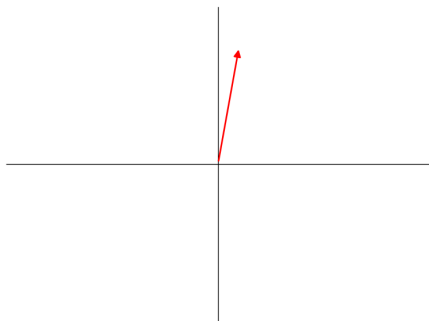
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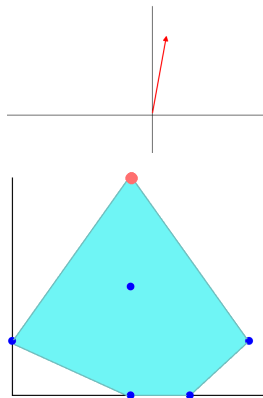
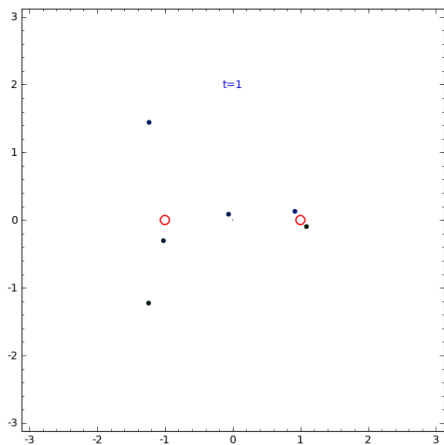
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As $t \mapsto \infty$, $V(f) \cap \mathcal{L}_t \mapsto V(42t^{24}\ell_1(s)^2\ell_2(s)^4) = \{1, 1, -1, -1, -1, -1\} \subset \mathbb{C}$

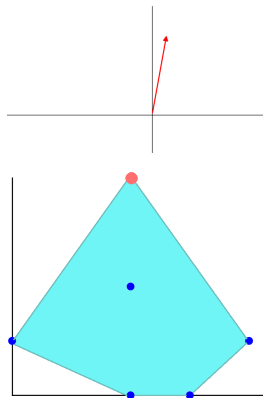
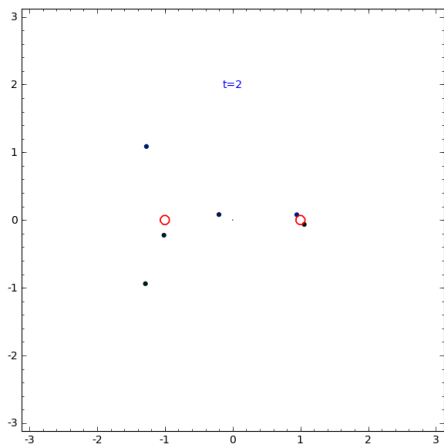
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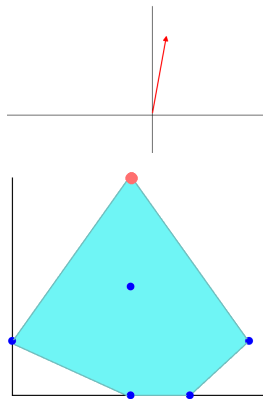
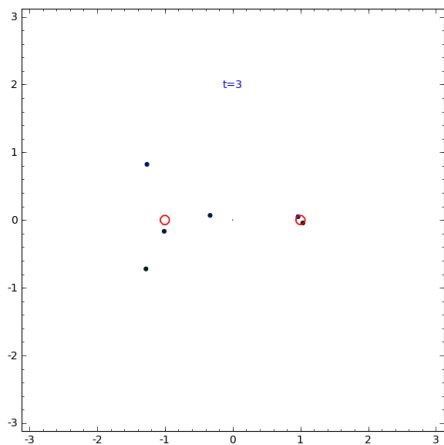
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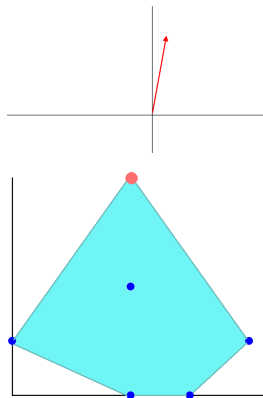
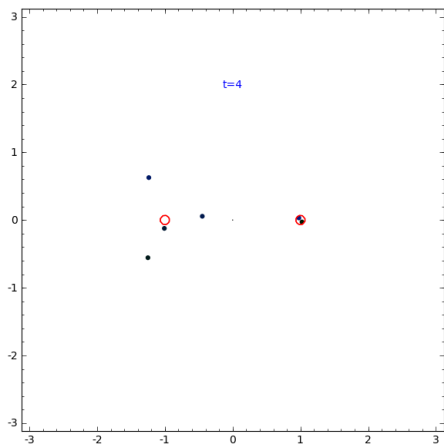
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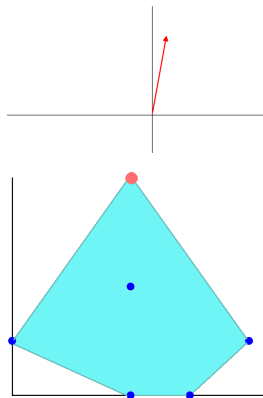
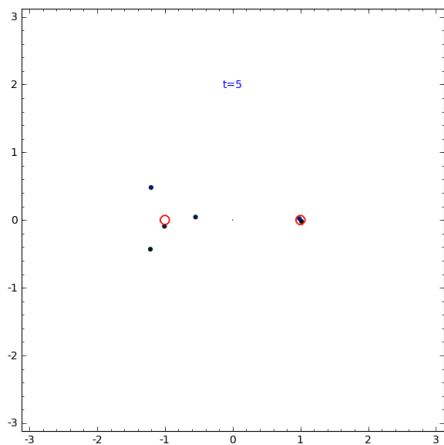
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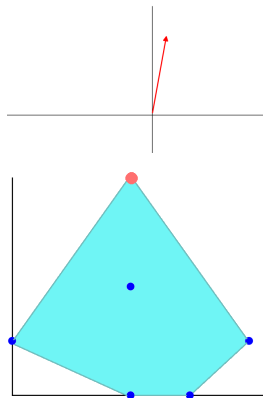
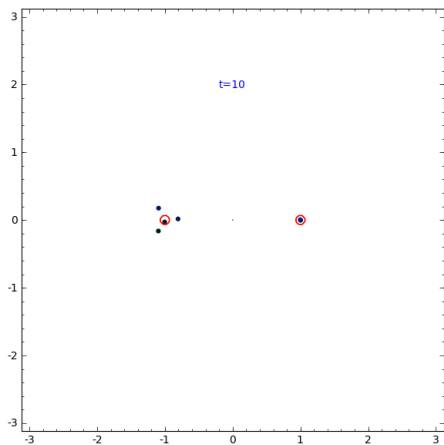
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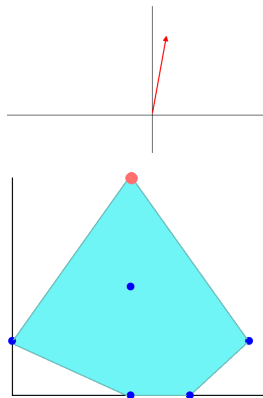
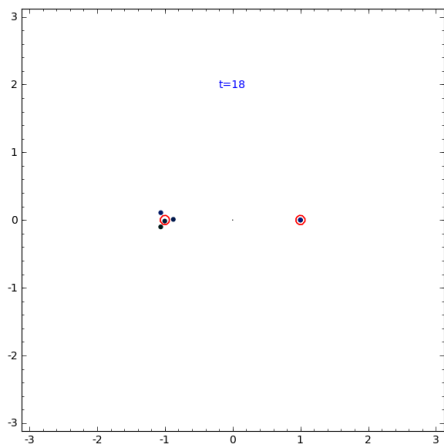
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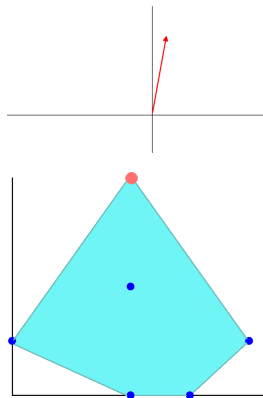
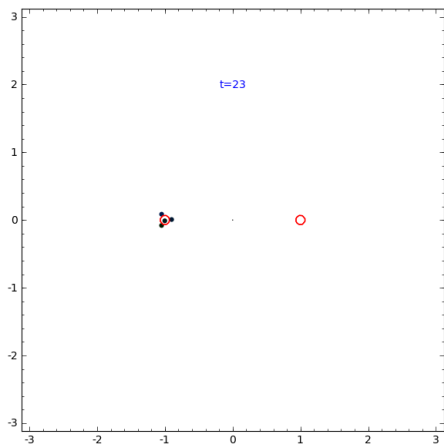
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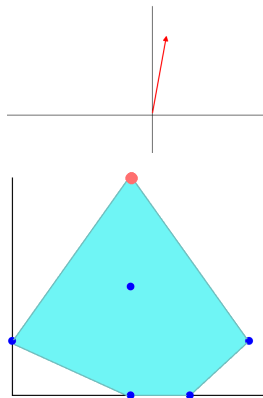
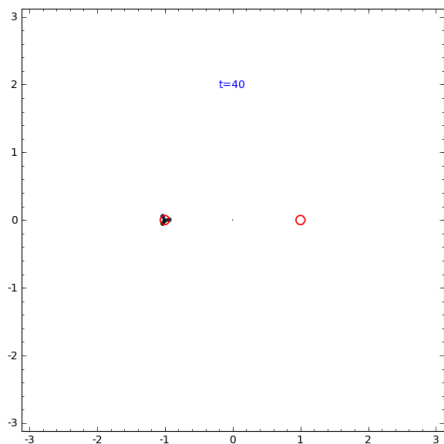
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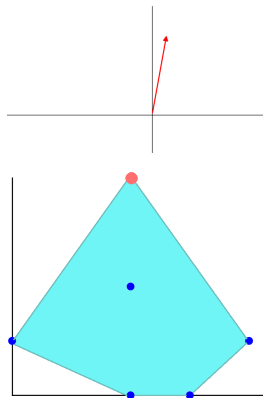
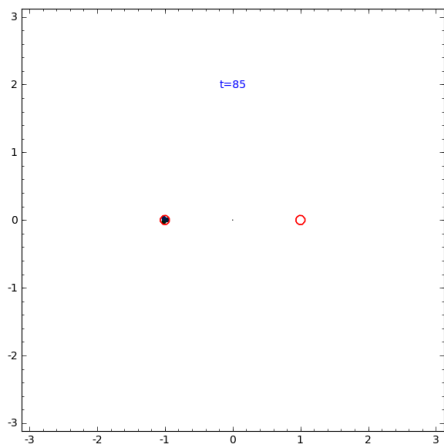
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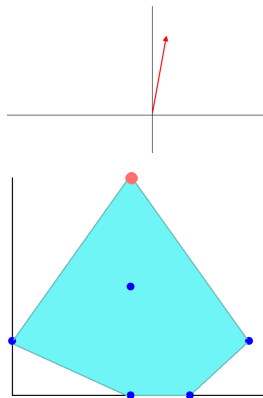
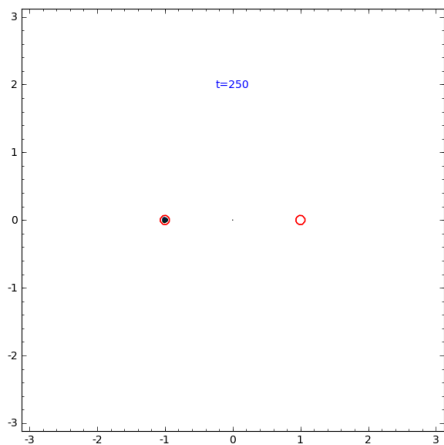
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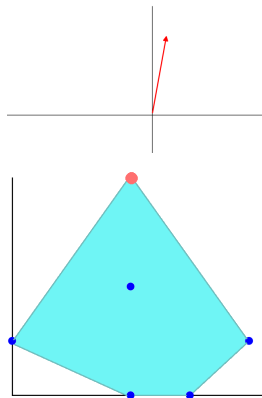
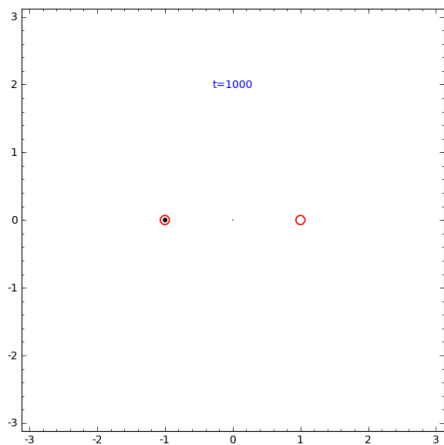
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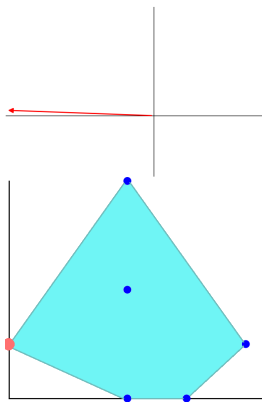
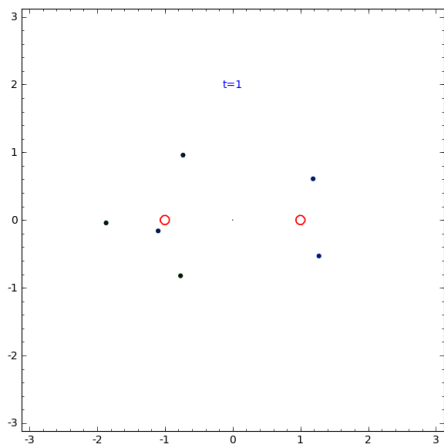
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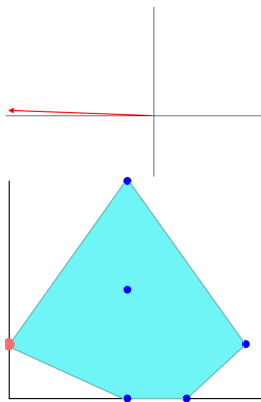
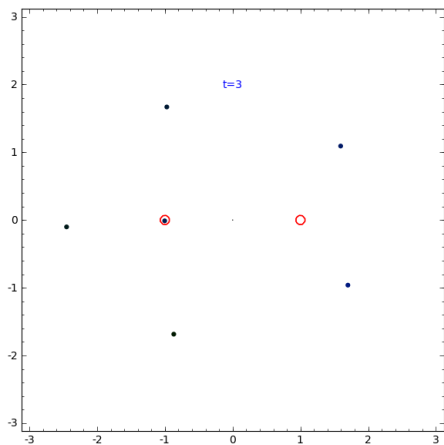
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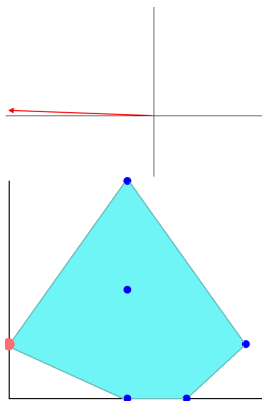
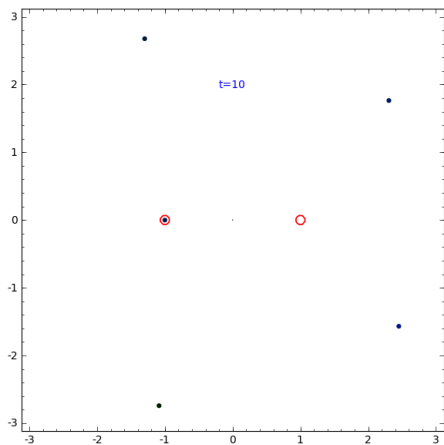
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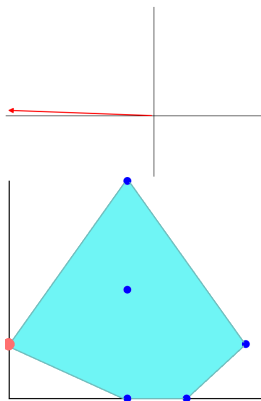
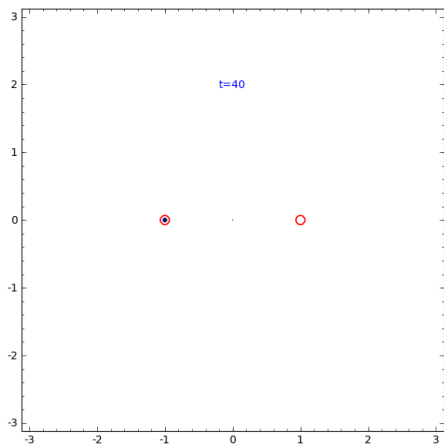
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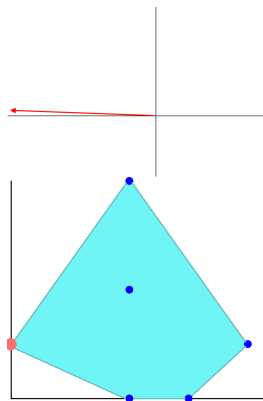
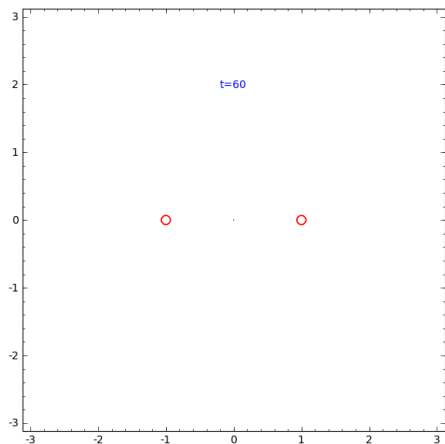
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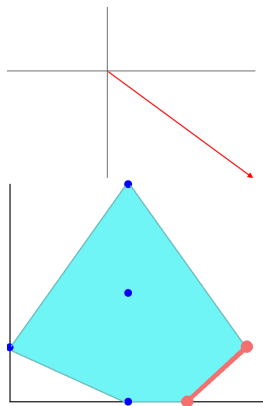
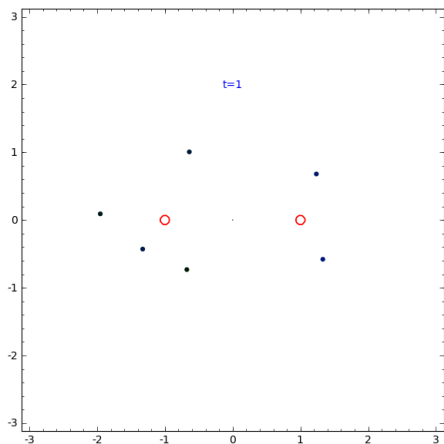
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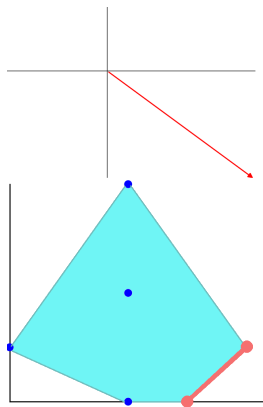
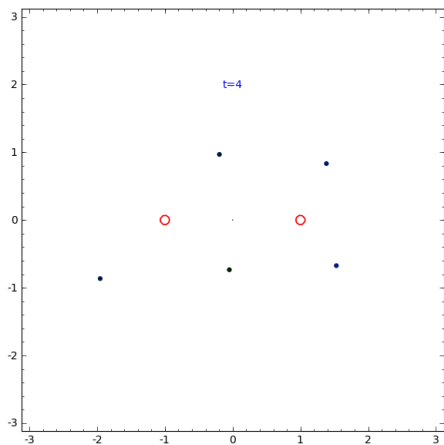
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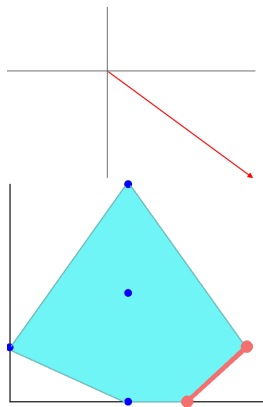
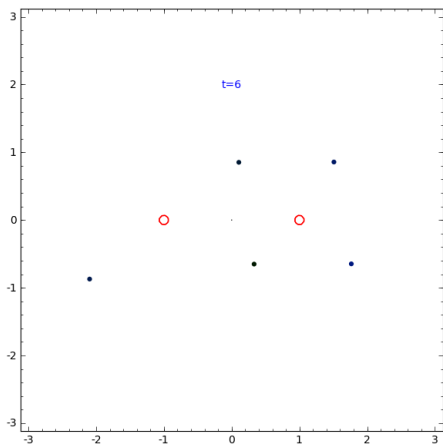
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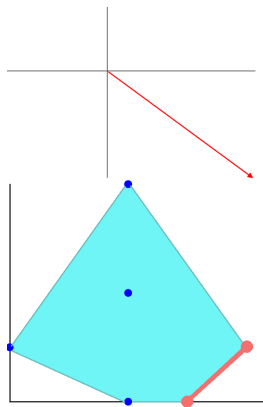
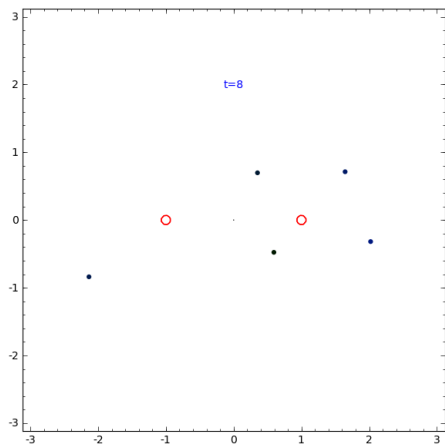
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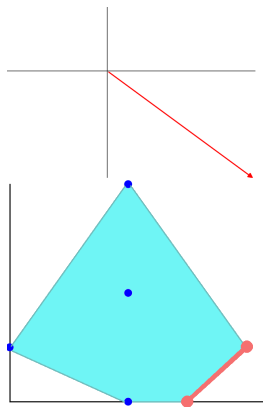
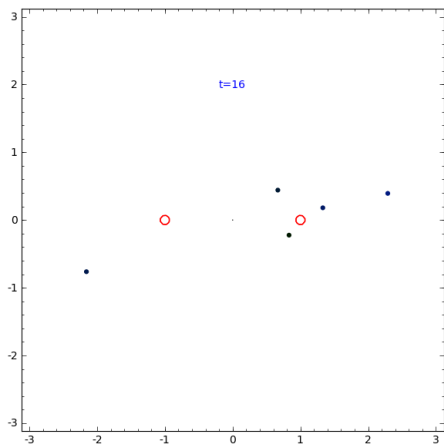
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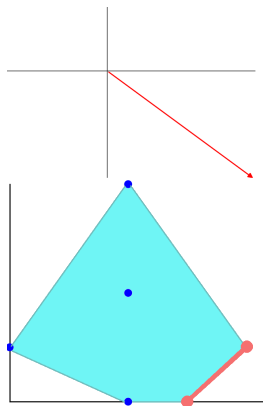
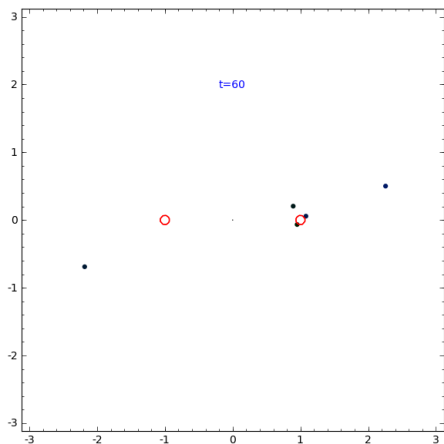
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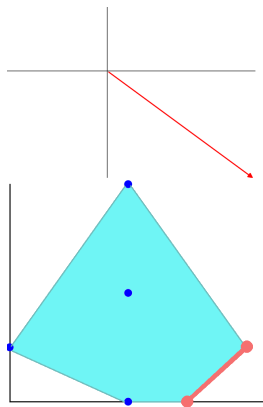
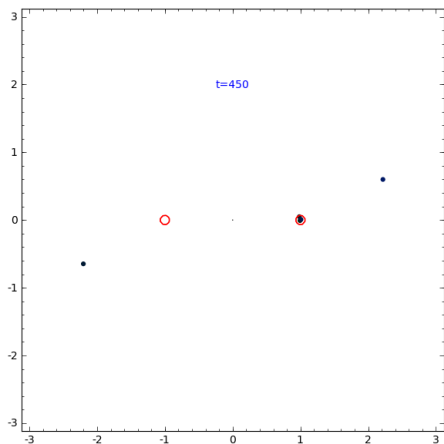
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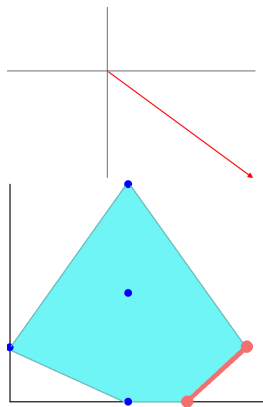
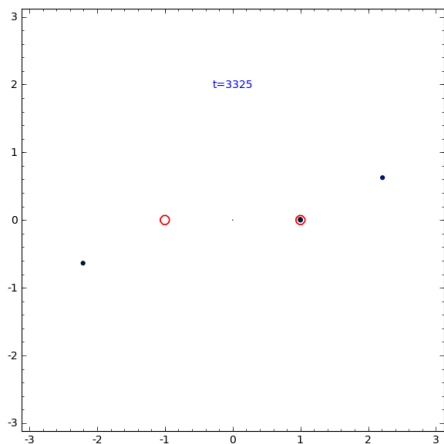
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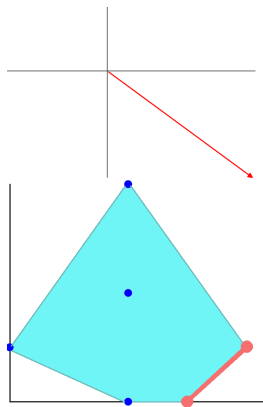
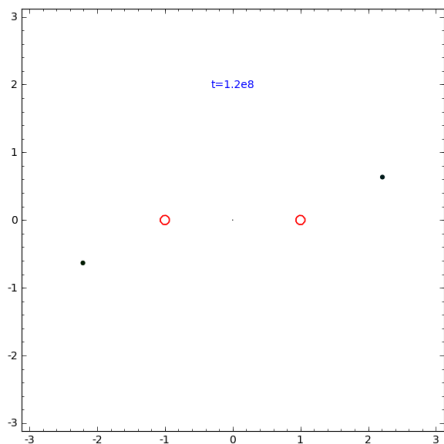
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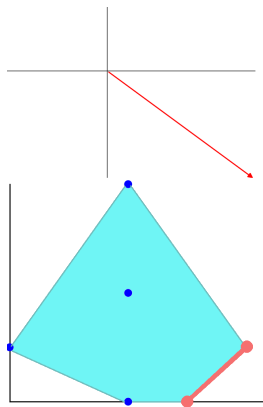
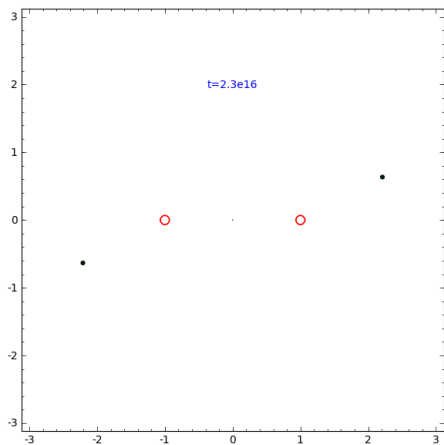
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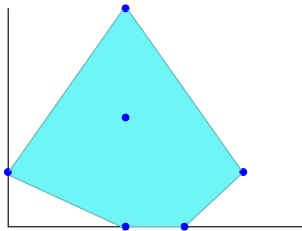
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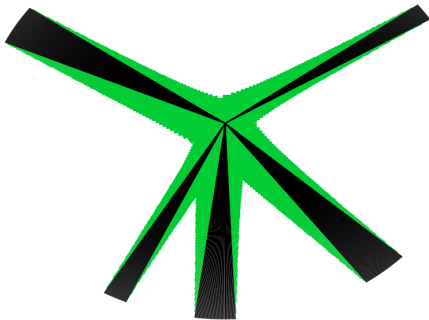
Runtimes \leftrightarrow Normal Fan and Numerical Issues

Direction close to exposing non-vertex \implies convergence happens slowly.

You can set how long you are willing to wait for convergence.



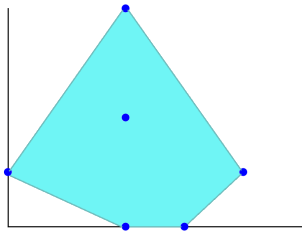
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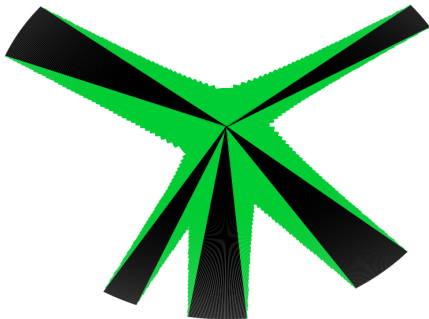
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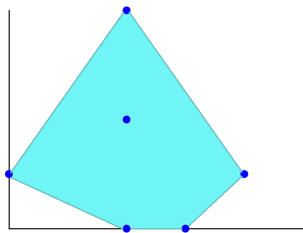
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$t \mapsto 1e9$



Bigger Examples: Reduced A-discriminants

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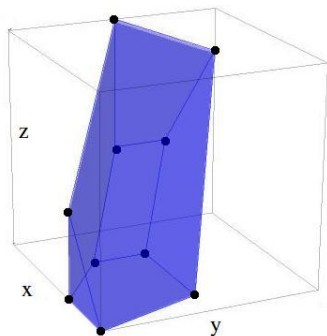
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We can produce witness sets for parametrizations and thus the algorithm applies.

Bigger Examples: A-Discriminants

Gale dual given by $B = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{bmatrix}$

the reduced \mathcal{A} -discriminant has the following Newton polytope



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 4 & 2 & 2 & 5 & 6 & 4 \\ 2 & 4 & 2 & 3 & 0 & 1 & 2 & 2 & 0 & 0 \\ 5 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Numerical Beneath-Beyond is currently work in progress.

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