Newton Polytopes via Numerical Algebraic Geometry

Taylor Brysiewicz (Texas A&M University)

October 8, 2016

(Work with Jonathan Hauenstein and Frank Sottile)

AMS sectional at University of Denver

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Given a hypersurface, get information about its defining polynomial.

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This talk: Algorithm by Hauenstein and Sottile [3]
Implementation in Macaulay2 using Bertini.m2 [1],[2].

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$$f(x, y) = 4y + 8x^2 + 15x^3 + 16x^2y^2 + 23x^4y + 42x^2y^4$$

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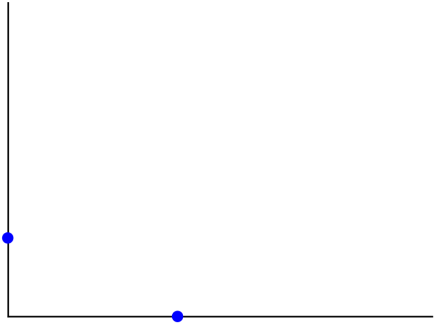

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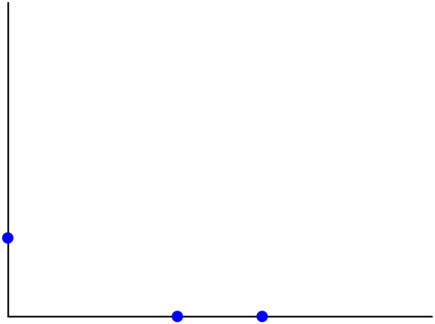

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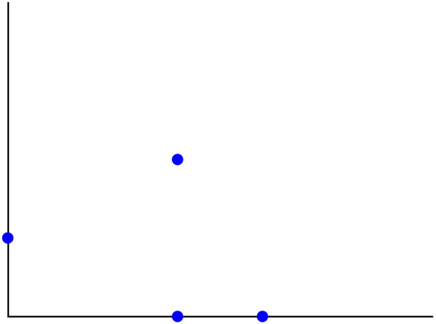

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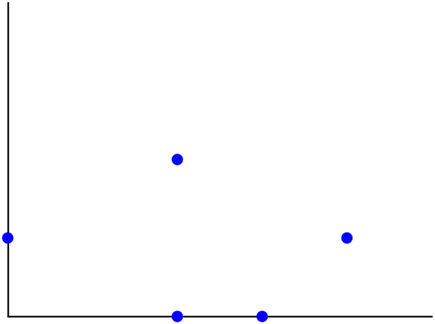

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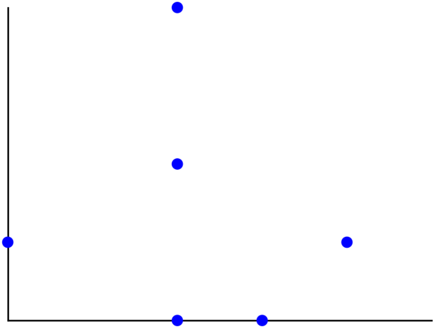

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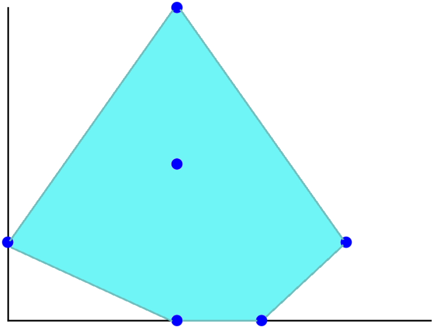

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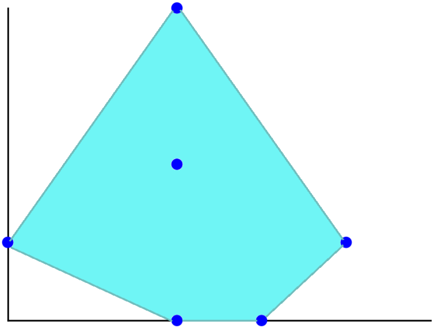

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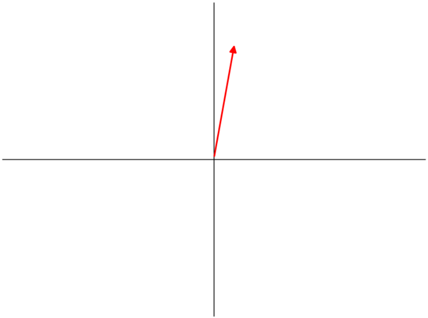

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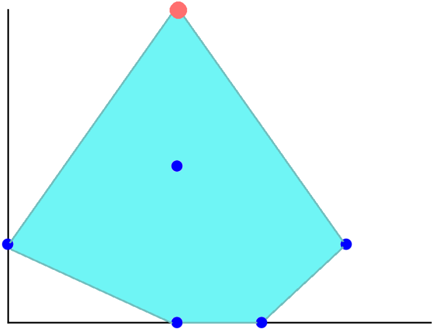

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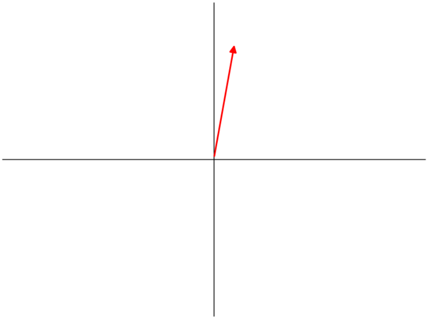

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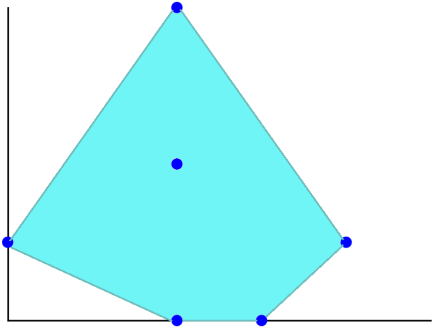

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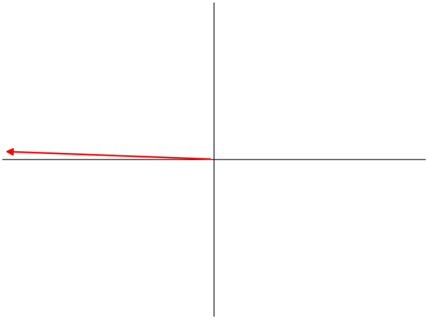

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$$f_{(-7,1)}(x,y) = 4y$$

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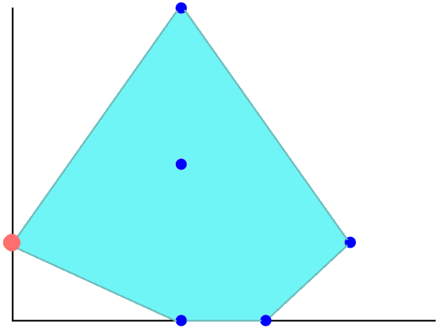

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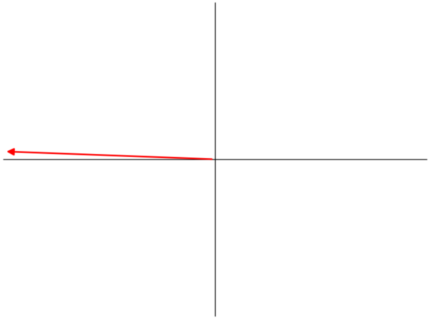

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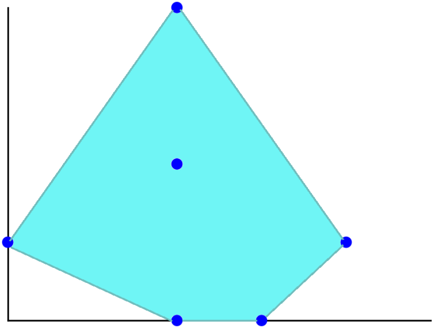

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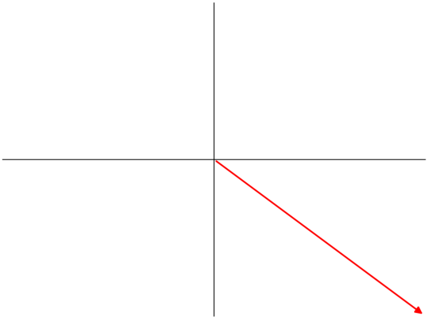

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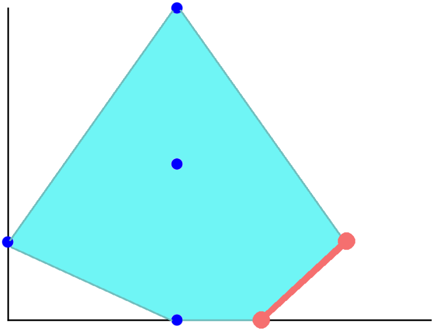

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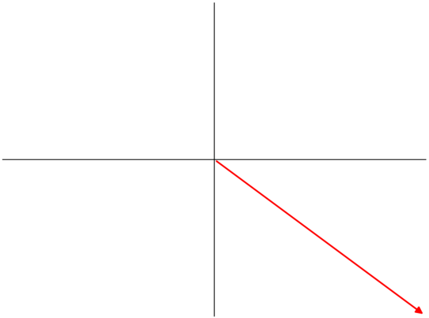

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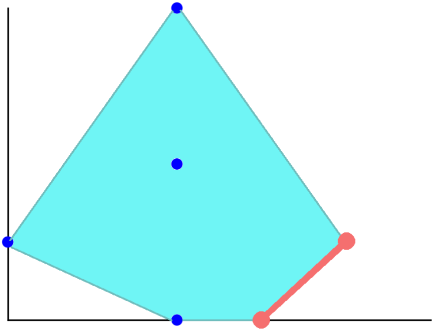

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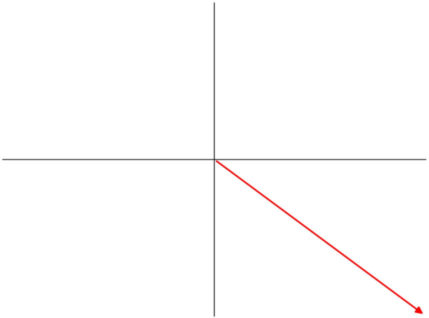

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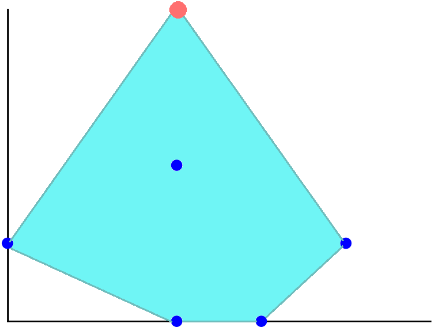

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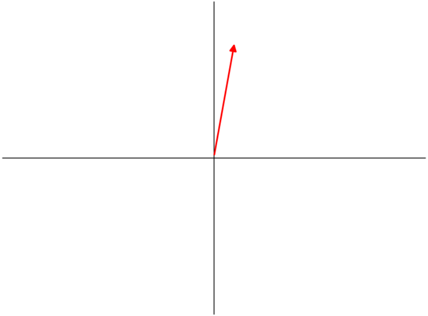

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As $t \mapsto \infty$, $V(f) \cap \mathcal{L}_t \mapsto V(42t^{24}\ell_1(s)^2\ell_2(s)^4) = \{1, 1, -1, -1, -1, -1\} \subset \mathbb{C}$

Demonstration

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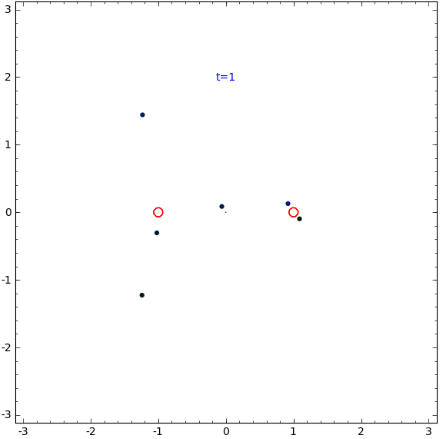

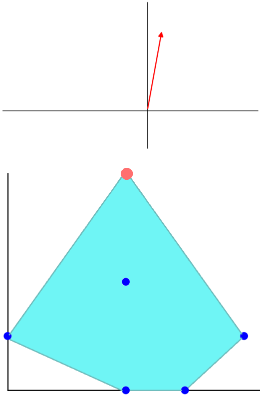

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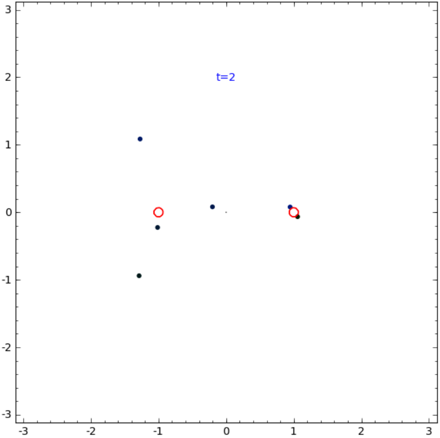

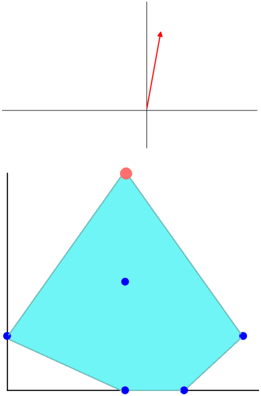

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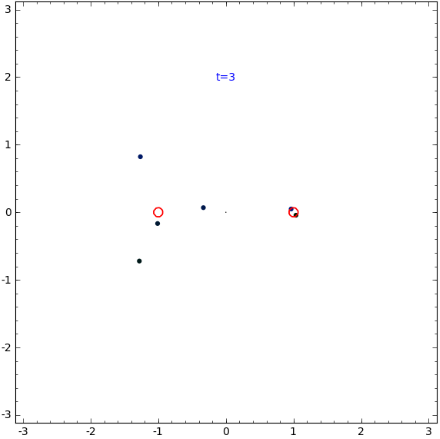

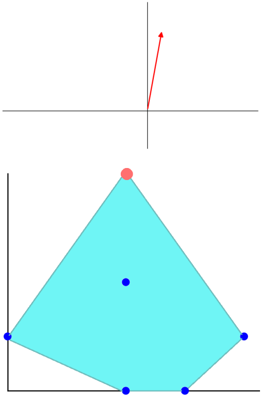

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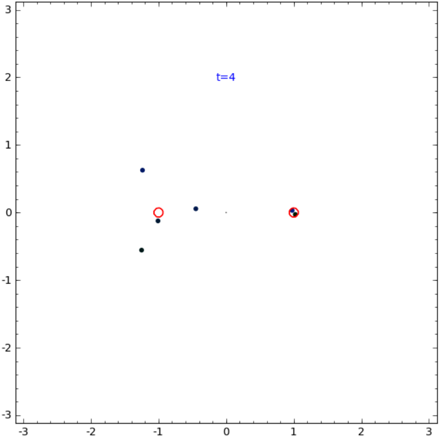

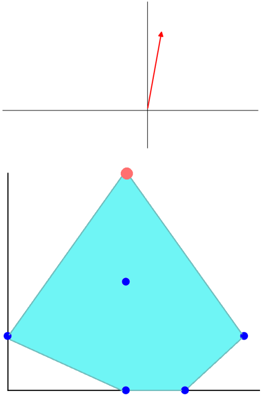

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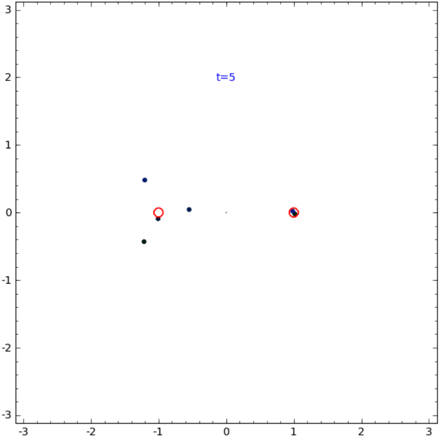

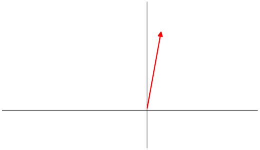

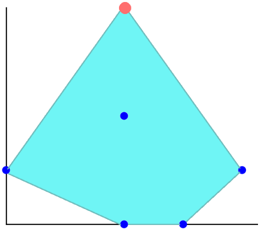

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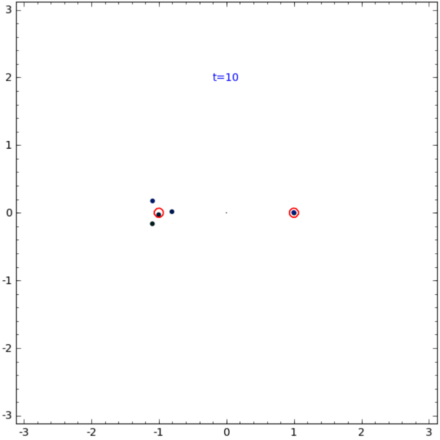

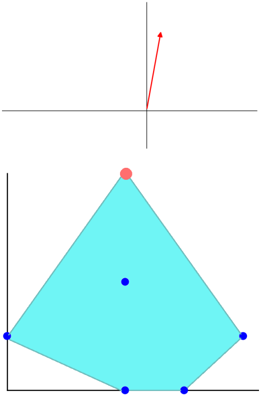

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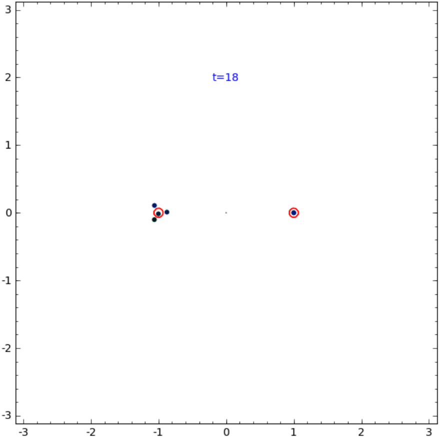

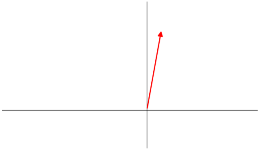

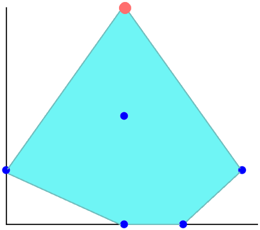

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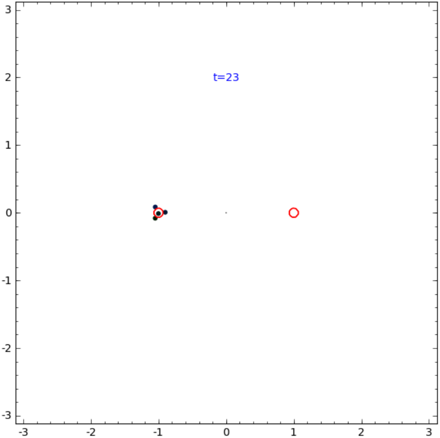

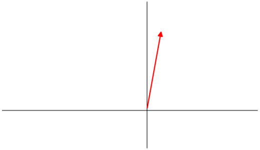

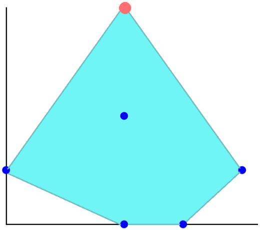

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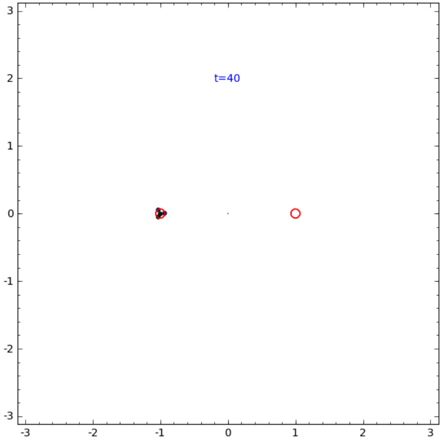

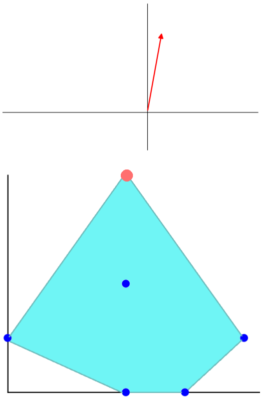

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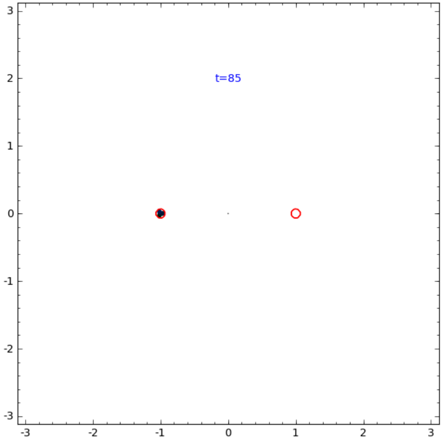

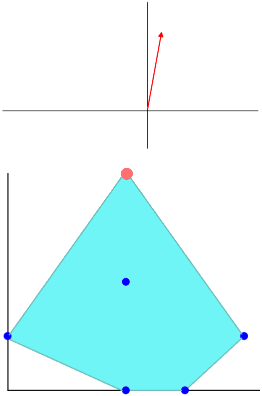

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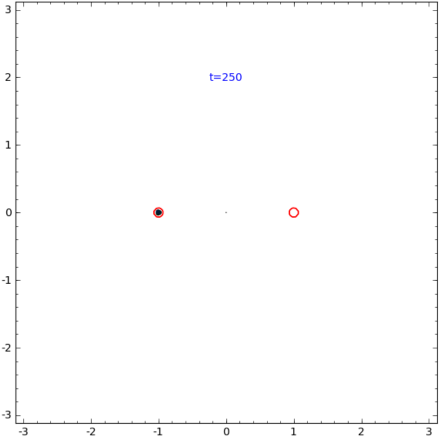

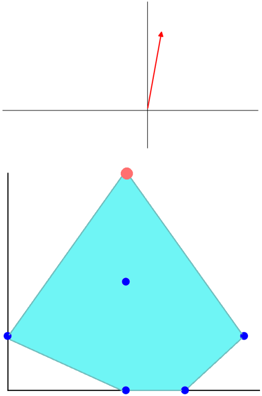

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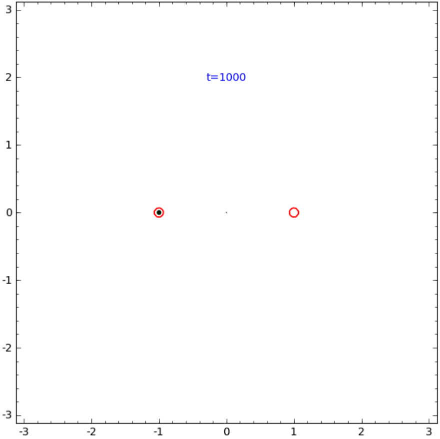

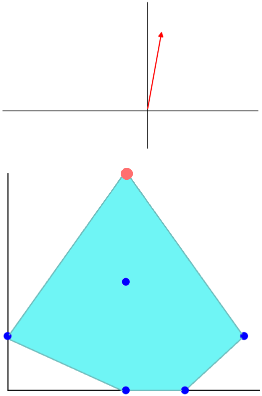

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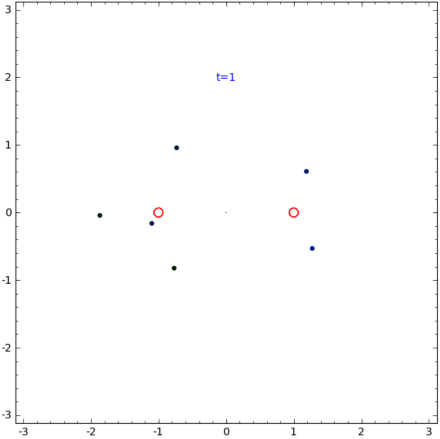

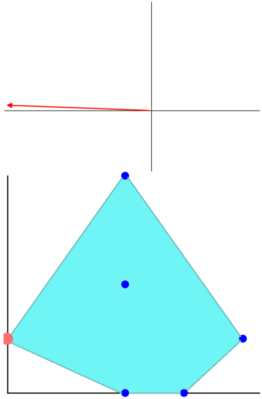

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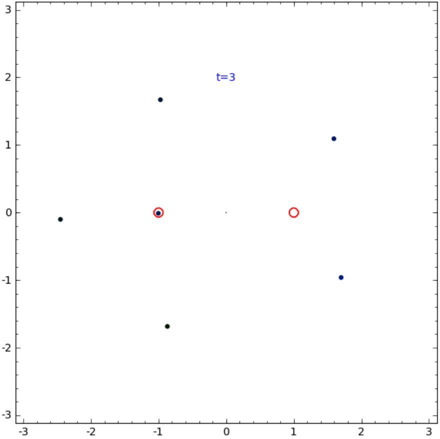

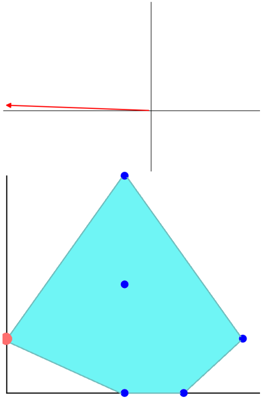

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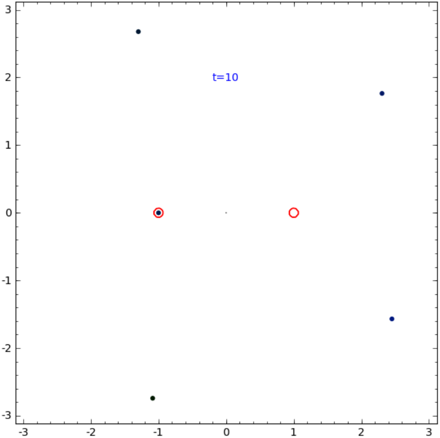

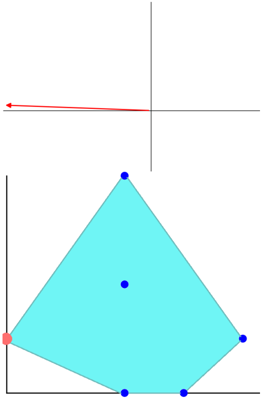

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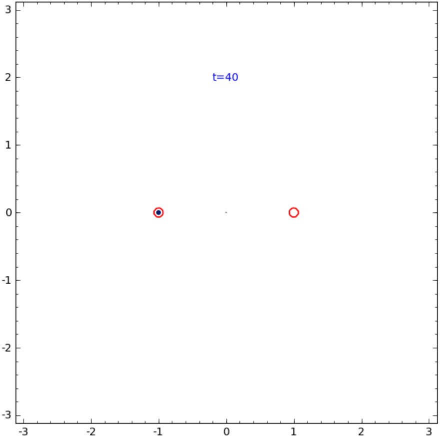

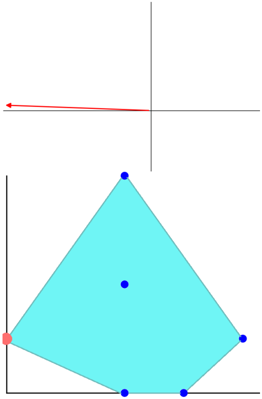

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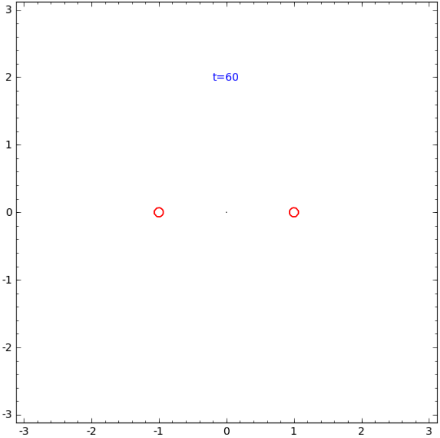

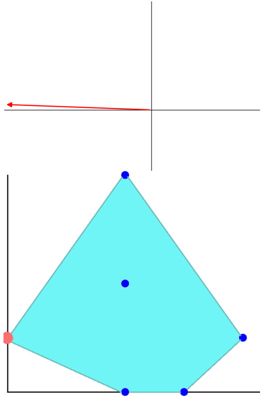

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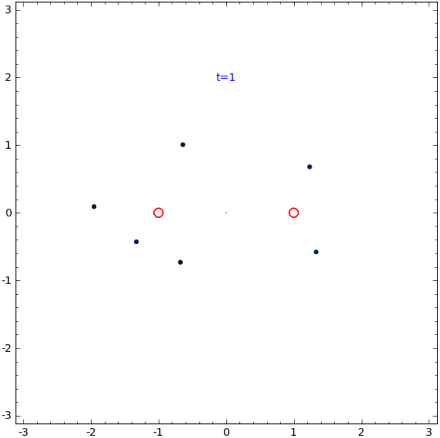

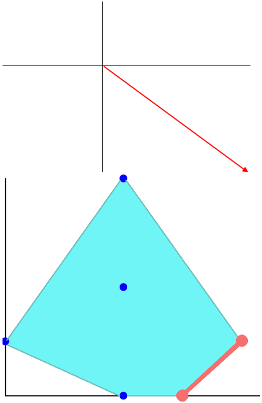

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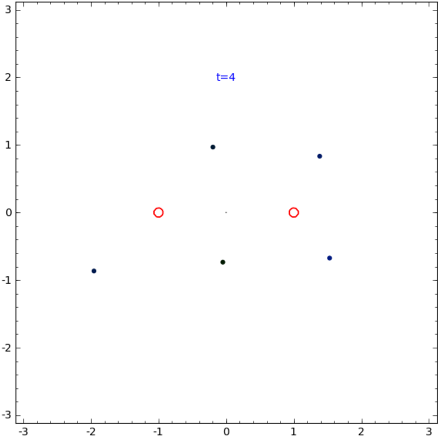

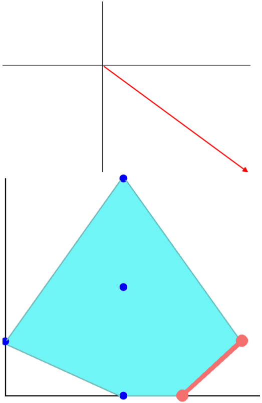

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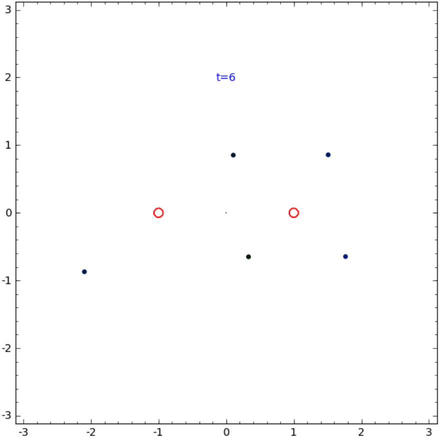

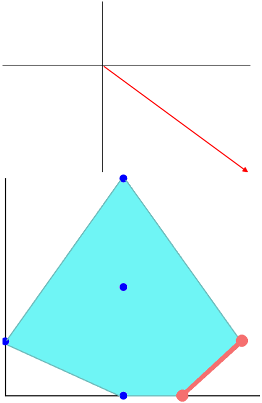

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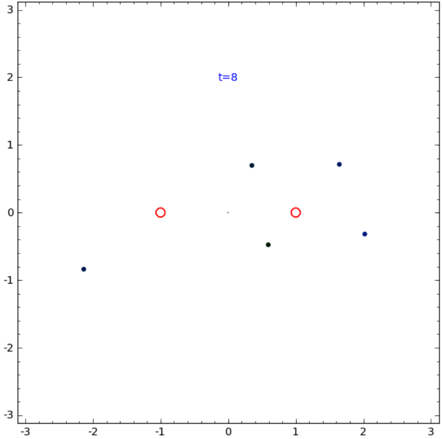

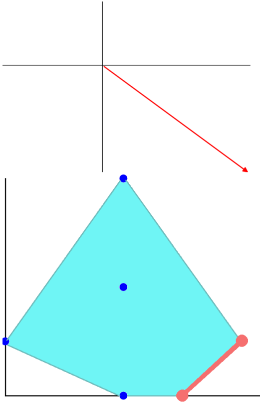

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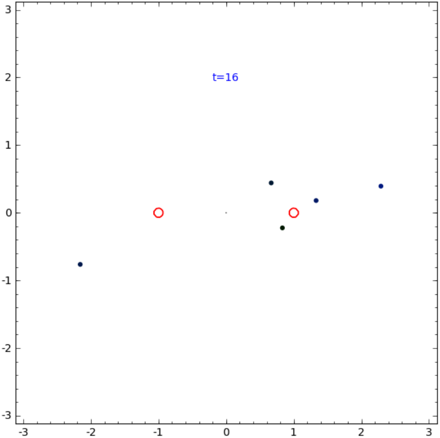

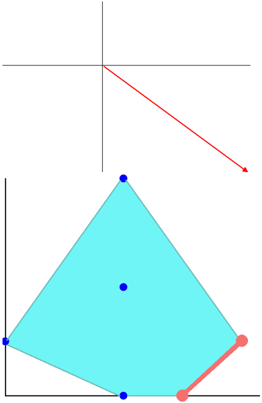

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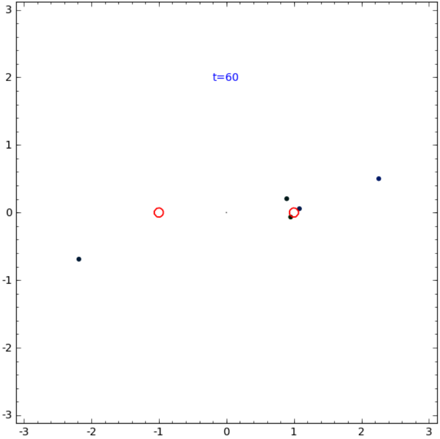

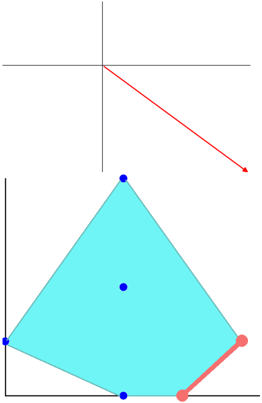

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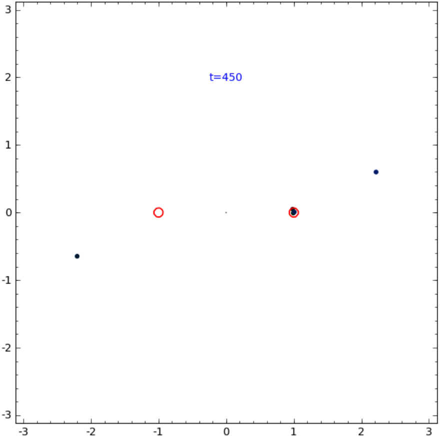

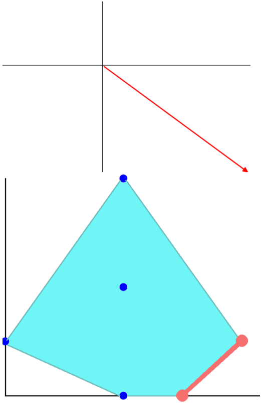

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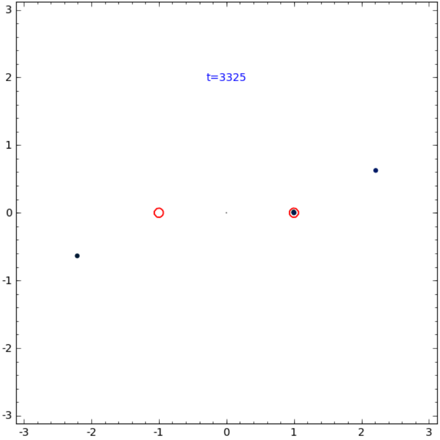

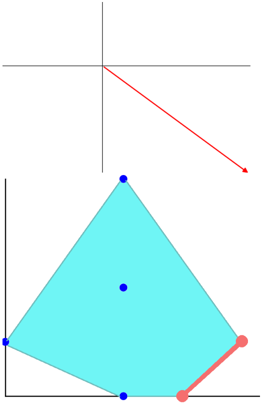

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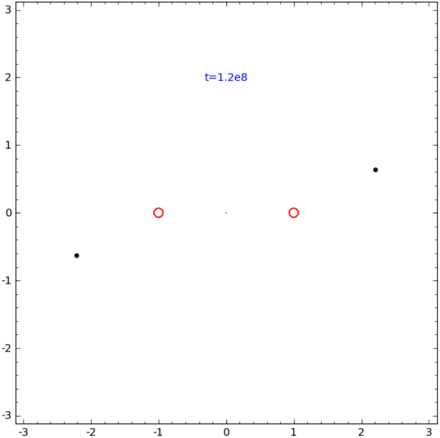

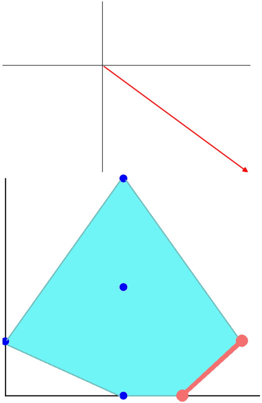

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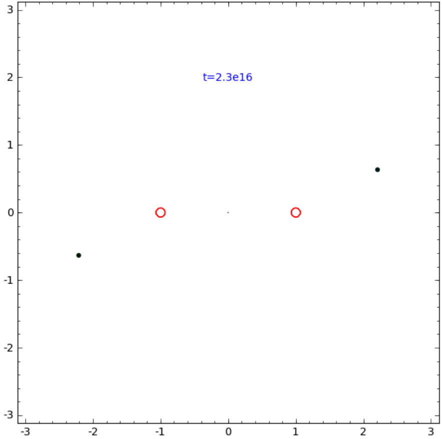
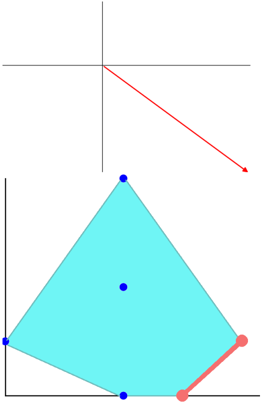


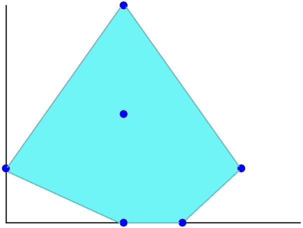


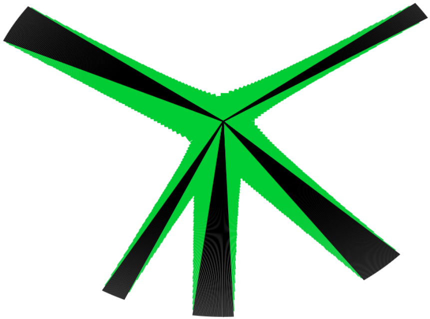
Runtimes $\leftrightarrow$ Normal Fan and Numerical Issues

Direction close to exposing non-vertex $\implies$ convergence happens slowly.

You can set how long you are willing to wait for convergence.

$t \mapsto 1e33$





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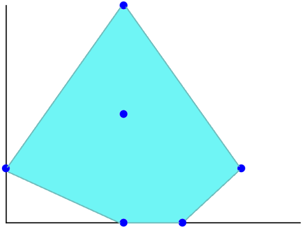

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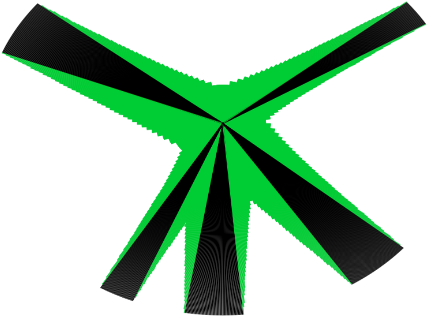

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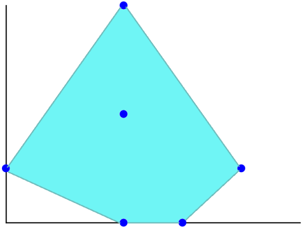

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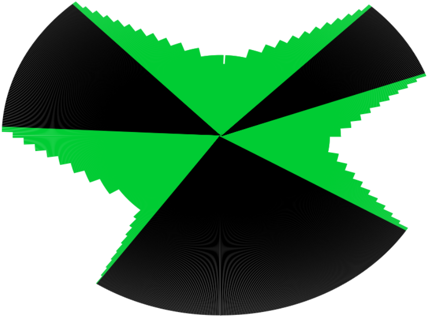

Bigger Examples: Reduced A-discriminants

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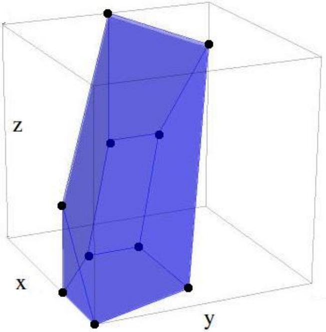
We can produce witness sets for parametrizations and thus the algorithm applies.

Bigger Examples: A-Discriminants

Gale dual given by $B = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{bmatrix}$
the reduced $\mathcal{A}$-discriminant has the following Newton polytope



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 4 & 2 & 2 & 5 & 6 & 4 \\ 2 & 4 & 2 & 3 & 0 & 1 & 2 & 2 & 0 & 0 \\ 5 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Numerical Beneath-Beyond is currently work in progress.

Works Cited

D. J. Bates, E. Gross, A. Leykin, and J. Israel Rodriguez.
Bertini for Macaulay2, October 2013.

Daniel J. Bates, Jonathan D. Hauenstein, Andrew J. Sommese, and Charles W. Wampler.
Bertini: Software for numerical algebraic geometry.

Jonathan D. Hauenstein and Frank Sottile.
Newton polytopes and witness sets.
Mathematics in Computer Science, 8(2):235–251, 2012.