Points of Ninth Order on a Cubic Curve

Cubics That Only Intersect Once

Taylor Brysiewicz
Northern Illinois University

Abstract

For any point on a plane cubic curve, it may or may not be possible to construct another cubic curve that will intersect it at that point with multiplicity nine, that is "completely". In our research, we provide a necessary and sufficient geometric condition for smooth points on a plane cubic curve to have this property. At such a point, one can construct an infinite family of these cubics and in particular, this family is parametrized by the osculating conic at that point.

Introduction

From Calculus, we know that the tangent line of a differentiable function at a point provides the best linear approximation for the function at that point. We then learned that the truncation of the power series of a function at a point gives the best n-th degree approximation of that function at that point.

If we no longer restrict ourselves to functions, but instead generalize to varieties in $\mathbb{C}^2$, we might ask what it means for two varieties to "approximate" each other well at a point. Furthermore, what does it mean to find the best n-th degree approximation of a variety at a point?

Intersection Multiplicity

Definition. *Given $f, g \in \mathbb{C}[x, y]$, and a point $p \in \mathbb{C}^2$, we define the* intersection multiplicity *of $\mathbf{V}(f)$ and $\mathbf{V}(g)$ (or just f and g) at p to be*

$$I_p(f, g) = \dim(\mathbb{C}[x, y]_p / \langle f, g \rangle)$$

where the dimension is taken as that of a $\mathbb{C}$-vector space.

Theorem (Bezout). *Let $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ be projective varieties of degrees m and n respectively. If C and D share no common components, then the number of points of intersection of C and D, counting multplicity, is mn.*

Question: When can two cubic curves intersect at only one distinct point, with multiplicity 9?

Osculating Curves

Definition. *Given an irreducible variety $C \subset \mathbb{P}^2_{\mathbb{C}}$ and a point $p \in C$, we define an* osculating curve *of n-th degree to C at p to be an n-th degree curve that provides the best approximation of C at p.*

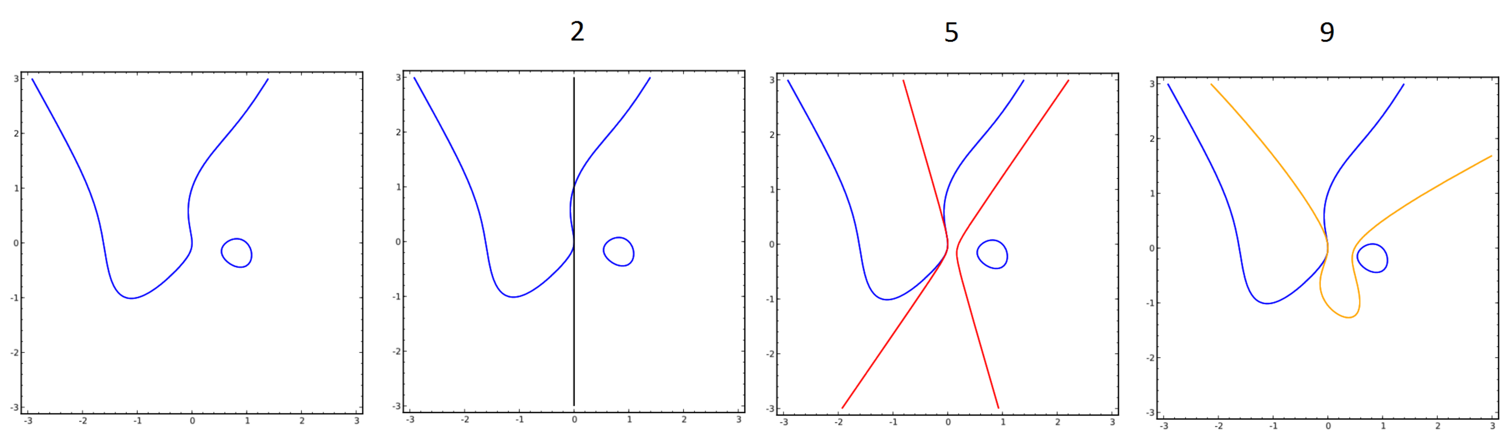
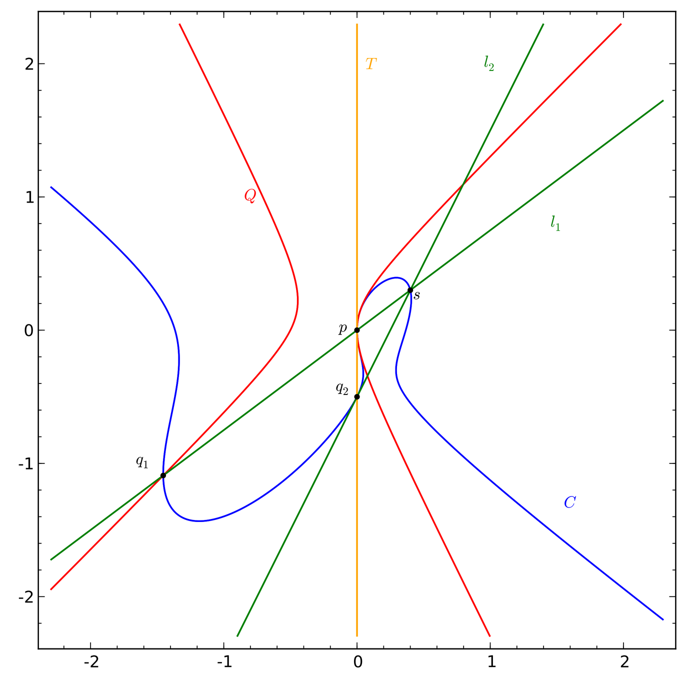


Figure 1: Osculating Curves

Rewriting Smooth Cubics

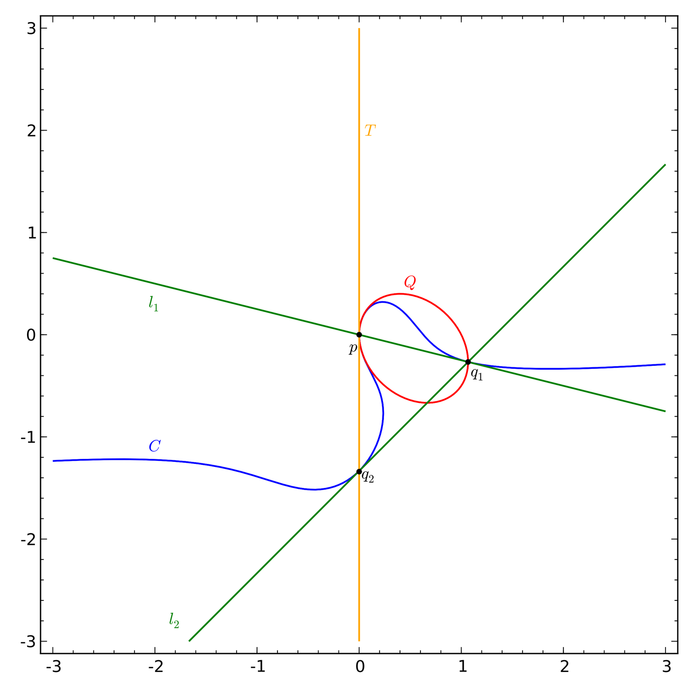
Theorem. *Let $\mathbf{V}(C) \subset \mathbb{C}^2$ be a smooth cubic through the non-flex point $p = (0, 0)$. We can always write C as $C = T^2 l_1 + Q l_2$.*



$$C = T^2 l_1 + Q l_2$$

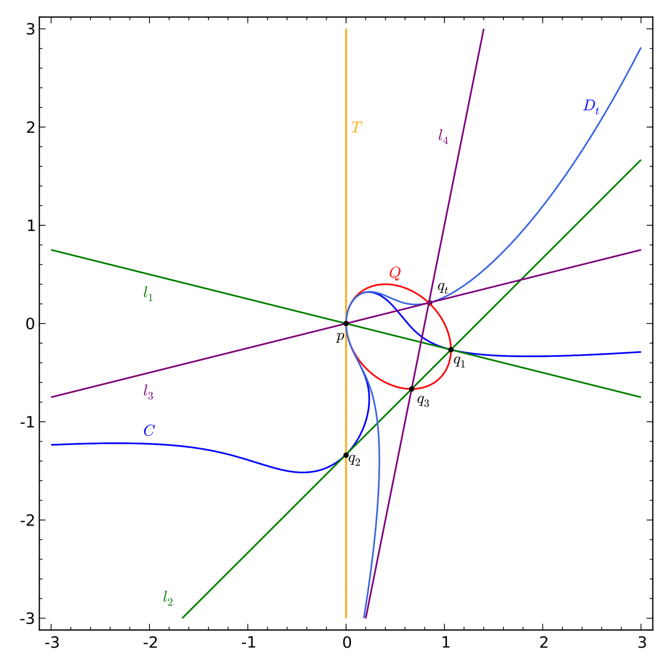
C- Cubic
Q- Osculating Conic of $\mathbf{V}(C)$ at p
T- Tangent Line of $\mathbf{V}(C)$ at p
q_1 - Sixth Point of $\mathbf{V}(C, Q)$
q_2 - Third Point of $\mathbf{V}(C, T)$
l_1- Line through p and q_1
l_2- Tangent Line of $\mathbf{V}(C)$ at q_2
s - $\mathbf{V}(l_1, l_2)$

Main Results



Theorem. *Given the smooth cubic $\mathbf{V}(C)$ with a non-flex at p, we can find another cubic to intersect it totally at p if and only if $s = q_1$.*

Remark. *If $\mathbf{V}(C)$ is an elliptic curve, this is equivalent to when p has order 9.*



Theorem. *Given the smooth cubic $\mathbf{V}(C)$ with a non-flex at p, if we can find another cubic to intersect it totally at p, then there are infinitely many such cubics. In particular, this family is in a one to one correspondence with points on $\mathbf{V}(Q)$.*

Corollary. *Precisely one of the cubics in this family is a singular cubic with a node at p. Under the correspondence mentioned before, this curve corresponds to the point p on the osculating conic $\mathbf{V}(Q)$.*

Acknowledgements

My research partner, Leah Balay-Wilson, from Smith College.
The professors at the University of Wisconsin - Stout REU program, in particular our advisor, Dr. Seth Dutter, for his incredible guidance throughout this project.
This research was supported in part by NSF REU grant DMS 1062403.