CHAPTER 14: Apportionment
14.1 The Apportionment Problem

An ***apportionment problem*** is to round a set of fractions so their sum is maintained at its original value.

The rounding procedure used in an apportionment problem is called an ***apportionment method***.

The total population, p, divided by the house size, h, is called the ***standard divisor, s***. $s = \frac{p}{h}$

A group's ***quota*** q_i is the group's population, p_i, divided by the standard divisor, s. $q_i = \frac{p_i}{s}$

Different apportionment methods will use different rounding rules.

When q is not already an integer, there are multiple ways to round.

- Round q up to the next integer, $\lceil q \rceil$.
- Round q down to the previous integer, $\lfloor q \rfloor$.
- Round to the nearest integer, $[q]$. If q is halfway to the next integer or larger, round up to the next integer. Otherwise, round down to the previous integer.
- Round according to the geometric mean. The geometric mean of $\lfloor q \rfloor$ and $\lceil q \rceil$ is $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$. If q is equal to or larger than q^*, round up to the next integer. Otherwise, round down to the previous integer.

<u>Example</u>
Complete the following chart.

q	$\lceil q \rceil$	$\lfloor q \rfloor$	$[q]$	q^*	Round according to q^*
5					
3.6					
3.5					
3.465					
3.464					
0.02					

<u>14.2 Hamilton Method</u>

Step 1 Compute the standard divisor.
Step 2 Compute the quota for each “state” (group).
Step 3 Round each quota *down*.
Step 4 Calculate the number of seats left to be assigned.
Step 5 Assign the remaining seats to the states with the *largest fractional part* of q.

Example

Use the Hamilton method to apportion 36 silver coins to Doris, Mildred, and Henrietta if Doris paid \$5900, Mildred paid \$7600, and Henrietta paid \$1400.

$s =$

Person	Contribution	q	Rounded quota	Hamilton Apportionment
Doris	\$5900			
Mildred	\$7600			
Henrietta	\$1400			
TOTAL				

Example

A county has four districts, North, South, East, and West. They will apportion for a 100 member advisory council using the Hamilton method. Determine the number of council members from each district.

$s =$

District	Population	q	Rounded quota	Hamilton Apportionment
North	27,460			
South	17,250			
East	19,210			
West	1000			
TOTAL				

14.3 and 14.4 Divisor Methods and Which Method is Best

We have used the standard divisor, *s*, to represent the average district population. We will use *s* for all apportionment methods to calculate the quota.

The divisor methods will also use an adjusted divisor, *d*, to calculate an adjusted quota. The adjusted quota combined with the appropriate rounding rules for each method will give the final apportionment for divisor methods.

Jefferson Method

Step 1 Compute the standard divisor.
Step 2 Compute the quota for each "state" (group).
Step 3 Round each quota *down*.
Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors, $d_i = \frac{p_i}{N_i+1}$, that correspond to giving each state one more seat.
Step 5 Assign a seat to the state with the *largest d*. (Notice that divisor methods look at the entire number of *d* rather than the fractional part of the number.)

Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, *d*.

Example

Let's return to the splitting of the 36 silver coins. Use the Jefferson method to distribute the coins.

$$s = \frac{14900}{36} \approx 413.8888889$$

Person	Cont.	q	Rounded quota	d_i	Jefferson App.
Doris	$5900	14.2550			
Mildred	$7600	18.3624			
Henrietta	$1400	3.3826			
TOTAL	**$14,900**				

$$d =$$

Example

When the ladies opened the bag of coins, they discovered that there were 37 coins. Use the Jefferson method to apportion the coins.

$s =$

Person	Cont.	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
D	$5900						
M	$7600						
H	$1400						
TOTAL	**$14,900**						

$d =$

Webster Method

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each "state" (group).

Step 3 Round each quota *to the nearest integer*.

Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors.

If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{N_i+0.5}$.

If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{N_i-0.5}$.

Step 5 Adjust the seats according to d.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, d.

Example

Let's return to the splitting of the 36 silver coins. Use the Webster method to distribute the coins.

$$s = \frac{14900}{36} \approx 413.8888889$$

Person	Cont.	q	Rounded quota	d_i	Webster App.
Doris	$5900	14.2550			
Mildred	$7600	18.3624			
Henrietta	$1400	3.3826			
TOTAL	**$14,900**				

$$d =$$

Example

Apportion the regions below using the Webster method for a house size of 16.

$s =$

Region	Pop.	q	Rounded quota	d_i	Webster App.
Beach	28,204				
Forest	11,267				
Plains	4,240				
Swamp	1,462				
TOTAL					

$d =$

Hill-Huntington Method

The Hill-Huntington method does a great job of keeping the relative differences of representative share $\left(\text{i.e., } \frac{\text{apportionment}}{\text{population}}\right)$ and district population $\left(\text{i.e., } \frac{\text{population}}{\text{apportionment}}\right)$ stable between states. It also ensures that every group gets at least one representative, so it favors small states. Since 1941, the Hill-Huntington method with a house size of 435 has been used to apportion the House of Representatives.

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each "state" (group).

Step 3 Round each quota *according to the geometric mean* of $\lfloor q \rfloor$ and $\lceil q \rceil$, $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$.

Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors.
If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{\sqrt{N_i(N_i+1)}}$.
If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{\sqrt{N_i(N_i-1)}}$.

Step 5 Adjust the seats according to d.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, d.

Example

Let's return to the splitting of the 36 silver coins. Use the Hill-Huntington method to distribute the coins.

$$s = \frac{14900}{36} \approx 413.8888889$$

Person	Cont.	q	q^*	Rounded quota	d_i	HH App.
Doris	$5900	14.2550				
Mildred	$7600	18.3624				
Henrietta	$1400	3.3826				
TOTAL	**$14,900**					

$d =$

Example

Apportion the regions below using the Hill-Huntington method for a house size of 16.

$$s = \frac{45{,}173}{16} = 2823.3125$$

Region	Pop.	q	q^*	Rounded quota	d_i	HH App.
Beach	28,204	9.990				
Forest	11,267	3.991				
Plains	4,240	1.502				
Swamp	1,462	0.518				
TOTAL	**45,173**					

$d =$

A ***paradox*** is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

Possible Issues - Alabama Paradox (Section 14.2)

The ***Alabama paradox*** occurs when a state loses a seat as the result of an increase in the house size.

Example

When you used the Hamilton method to apportion 36 silver coins to the ladies, Doris received 14, Mildred received 18, and Henrietta received 4 coins. Assume that they found an extra coin and use the Hamilton method to apportion 37 silver coins to the ladies.

$$s = \frac{14{,}900}{37} \approx 402.703$$

Person	Contribution	q	Rounded quota	Hamilton Apportionment
Doris	\$5900	14.6510		
Mildred	\$7600	18.8725		
Henrietta	\$1400	3.4765		
TOTAL	**\$14,900**			

What information tells you that the Alabama paradox occurred in this example?

Possible Issues - Population Paradox (Section 14.2)

Consider two numbers, A and B, where $A > B$.
The ***absolute difference*** between the two numbers is $A - B$

The ***relative difference*** between the two numbers is $\dfrac{A-B}{B} \times 100\%$

The ***population paradox*** occurs when there are a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Example

Earlier, we apportioned 100 council members to four districts. Initially, North had 42 seats, South had 27 seats, East had 30 seats, and West had 1 seat. Ten years later, the county reapportioned the 100 council seats using new population data. Use the Hamilton method for this apportionment

$s =$

District	Population	q	Rounded quota	Hamilton Apportionment
North	28,140			
South	17,450			
East	19,330			
West	990			
TOTAL				

Compare the districts' populations.

District	Initial Pop.	New Pop.	Absolute Difference	Relative Difference
North	27,460	28,140		
South	17,250	17,450		
East	19,210	19,330		
West	1000	990		

Did the population paradox occur?

Explain what information helped you determine whether or not the population paradox occurred.

Possible Issues – New States Paradox (Section 14.2)

The ***new states paradox*** occurs in a reapportionment in which an increase in the total number of states (with a proportionate increase in representatives) causes a shift in the apportionment of existing states.

Example

A country has two states, Solid and Liquid. Use Hamilton's method to apportion 12 seats for their congress

s =

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899			
Liquid	59,096			
TOTAL				

Another state, Plasma, wants to join. If there are 38,240 people in that state, how many representatives should they receive?

Use Hamilton’s method to apportion the seats for their congress (the 12 original seats plus the additional seats that were added when Plasma joined).

s =

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899			
Liquid	59,096			
Plasma	38,240			
TOTAL				

What information tells you that the new states paradox occurred in this example?

Possible Issues – Quota Condition (Section 14.3)

Example

A school offers four different art classes with the enrollments shown below. Ten new teachers will be hired according to an apportionment using Jefferson's method. Determine who gets the new teachers.

$s =$

Class	Enrollment	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
Ceramics	785						
Painting	152						
Dance	160						
Theatre	95						
TOTAL							

The ***quota condition*** says that the number assigned to each represented unit must be the standard quota, q, rounded up or rounded down.

What information tells you that the quota condition was violated in this example?

Comparing Methods

Balinski and Young found that no apportionment method that satisfies the quota condition is free of paradoxes.

- Divisor methods are free of the paradoxes, but they can violate the quota condition.
- Hamilton's method may have paradoxes but does not violate the quota condition.

Sample Exam questions

Sample exam questions are likely to focus on performing all four apportionment methods and recognizing each of the four issues (three paradoxes and the quota condition).