*Homework for Math 689, Polynomials and Polynomial Inequalities*
*Fall 2013, MWF 01:50 – 02:40 pm, BLOC 684*

**Homework:**

Please, do each part of each problem unless it is stated otherwise.
• Group 0: Make sure you know how to do these exercises. These results are closely related to the topics discussed in lectures and they may be used later in proofs presented in the lectures later.
• Group 1: These results are also basic with rather straightforward solutions. You should know these results even if you may not wish to convince yourself with a rigorous proof.
• Group 2: More subtle problems with copious hips or complete outlines. You may learn a lot from making the arguments completely transparent for yourself. Most likely these results may not be needed in any arguments presented in the lectures later.
• Group 3: I do not know how to do these, and perhaps nobody knows. If you make significant progress in these with correct proofs, your work will almost certainly be published in a prestigious research journal. (You may or may not wish to waste your time on thinking about these too much.)

- Group 0

- Section 1.1: 3, 6, 7, 8, 9
- Section 1.2: 1, 2, 3, 4.
- Section 1.3: Read the proof of Lucas' Theorem (Theorem 1.3.1).
- Section 2.1: 1, 2, 3, 4, 7, 8, 10.
- Section 2.2: 1, 2, 3, 4, 5.
- Section 2.3: 1, 2.
- Section 2.4: -
- Section 3.1: 1, 4, 5, 6, 8, 10, 11, 12
- Section 3.3: 1, 2, 3
- Section 4.2: 3, 4
- Section 5.1: Read the proof of the Remez inequality (Theorem 5.1.1 on p. 228). You may be interested in reading the complete proof of the trigonometric Remez inequality in http://jlms.oxfordjournals.org/content/s2-45/2/255.full.pdf
- Section 5.1: 1, 3, 4, 5, 6.
- Section 6.2: E1 a], b] c] d] e], E8 a], b] c] d] e].
- Section 3.5 (pages 139-153).
- Section 7.1 (pages 320-331). Do as many of E 1-12 as you like (pages 332-344).

- Group 1

- Section 1.1: 10.
- Section 1.2: -
- Section 1.3: 1, 6.
- 11.

- Section 2.2: 6, 7, 8, 9.
- Section 2.3: 8, 13, 18.
- Section 2.4: 2, 3.
- Appendix 3: Please, read it.

- Group 2

- Section 1.1: 2.
- Section 1.2: 7.
- Section 1.3: 1, 12, 13.
- Section 2.1: 6.
- Section 2.2: -
- Section 2.3: 9, 10, 11, 12, 14, 15, 16 a], b].
- Section 2.4: 4, 5.

- Group 3

Notation:

$$\mathcal{F}_n := \left\{ p : p(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \{-1, 0, 1\} \right\}$$

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \{-1, 1\} \right\}$$

$$\mathcal{K}_n := \left\{ p : p(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}, \quad |a_j| = 1 \right\}$$

$$\mathcal{A}_n := \left\{ p : p(z) = \sum_{j=1}^{n} z^{\lambda_j}, \qquad 0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n, \quad \lambda_j \in \mathbb{Z} \right\}$$

$$\mathcal{B}_n := \left\{ p : p(t) = \sum_{j=0}^{n} \cos(\lambda_j t), \qquad 0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n, \quad \lambda_j \in \mathbb{Z} \right\}$$

$$M_0(Q) := \exp\left( \frac{1}{2\pi} \int_0^{2\pi} \log |Q(e^{it})| \, dt \right)$$

$$M_p(Q) := \left( \frac{1}{2\pi} \int_0^{2\pi} \left|Q(e^{it})\right|^p \, dt \right)^{1/p}, \qquad p > 0$$

$$M_\infty(Q) := \max_{t \in \mathbb{R}} |Q(e^{it})|$$

- P1. Is it true that there are $0 \not\equiv p_n \in \mathcal{F}_n$ having at least $c\sqrt{n}$ zeros at 1 with an absolute constant $c > 0$?

• P2. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{L}_n$ has at most $c \log n$ zeros at 1?

• P3. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{K}_n$ has at most $c(\log n)^2$ zeros at 1?

• P4. Is there an absolute constant $\epsilon > 0$ such that

$$M_\infty(p) \geq (1+\varepsilon)\sqrt{n+1}$$

for every $p \in \mathcal{L}_n$?

• P5. Is there an absolute constant $\epsilon > 0$ such that

$$M_\infty(p) \geq \sqrt{n+1} + \varepsilon$$

for every $p \in \mathcal{L}_n$?

• P6. Is there an absolute constant $\epsilon > 0$ such that

$$M_4(p) \geq (1+\varepsilon)\sqrt{n+1}$$

for every $p \in \mathcal{L}_n$?

• P7. Is there an absolute constant $\epsilon > 0$ such that

$$M_4(p) \geq \sqrt{n+1} + \varepsilon$$

for every $p \in \mathcal{L}_n$?

• P8. Is there absolute constant $c > 0$ such that every $p \in \mathcal{L}_n$ has a zero in the annulus

$$\{z \in \mathbb{C} : 1 - c/n < |z| < 1 + c/n\}?$$

• P9. Is there a sequence $(p_n)$ of Littlewood polynomiald $p_n \in \mathcal{L}_n$ such that

$$\min_{z \in \partial D} |p_n(z)| \geq c\sqrt{n}$$

holds with an absolute constant $c > 0$?

• P10. Is there a sequence $(p_{n_k})$ of Littlewood polynomiald $p_{n+k} \in \mathcal{L}_{n_k}$ such that

$$\min_{z \in \partial D} |p_{n_k}(z)| \geq c\sqrt{n_k}, \qquad n_k \to \infty,$$

holds with an absolute constant $c > 0$?

• P11. Is there a sequence $(p_n)$ of ultraflat Littlewood polynomials $p_n \in \mathcal{L}_n$ such that

$$\lim_{n\to\infty} \max_{z\in\partial D} \left| \frac{|p_n(z)|}{\sqrt{n}} - 1 \right| = 0\,?$$

• P12. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{L}_n$ has at most $c(\log n)^2$ real zeros?

• P13. Is it true that for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ and $p \in \mathcal{F}_n$ such that $p(0) = 1$ and $p$ has a zero in $D \setminus \{0\}$ with multiplicity at least $k$?

• P14. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{B}_n$ has at least $c \log n$ zeros in $[0, 2\pi)$ for all sufficiently large $n \in \mathbb{N}$?

• P15. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{B}_n$ has at least $c \log\log n$ zeros in $[0, 2\pi)$ for all sufficiently large $n \in \mathbb{N}$?

• P16. Is there a sequence $(p_{n_k})$ of Littlewood polynomials $p_{n_k} \in \mathcal{L}_{n_k}$ such that

$$\lim_{k\to\infty} \frac{M_0(p_{n_k})}{\sqrt{n_k}} = 1\,?$$

• P17. Let $N(p)$ denote the number of zeros of $p$ in the closed unit disk $\overline{D}$? Is it true that $N(p_n) \to \infty$ whenever $p_n \in \mathcal{L}_n$?

• P18. Let $p$ be a prime and let $f_p$ be the $p$-th Fekete polynomial. Is if true that

$$\frac{M_\infty(f_p)}{\sqrt{p}\log\log p}$$

can be arbitrary large?

• P19. Let $p$ be a prime and let $f_p$ be the $p$-th Fekete polynomial. Is it true that $M_\infty(f_p) = o(\sqrt{p}\log p)$? can be arbitrary large?

• P20. Is it true that there is an absolute constant $\varepsilon > 0$ such that

$$M_\infty(\mathrm{Re}(p)) \geq (1+\varepsilon)\sqrt{n}$$

for every $p \in \mathcal{L}_n$?

• P21. Does Müntz's theorem hold on the ternary Cantor set?

• P22. Is there a combact set $A \subset (0, \infty)$ with Lebesgue measure $m(A) = 0$ on which Müntz's theorem holds?

• P23. Is there a sequence $(p_{n_k})$ with $p_{n_k} \in \mathcal{A}_{n_k}$ such that

$$\lim_{k\to\infty} \frac{M_1(p_{n_k})}{\sqrt{n_k}} = 1\,?$$

• P24. Is there a sequence $(p_{n_k})$ with $p_{n_k} \in \mathcal{A}_{n_k}$ such that

$$\lim_{k\to\infty} \frac{M_1(p_{n_k})}{\sqrt{n_k}} > \sqrt{3}/2\,?$$

• P25. Is there a sequence $(p_{n_k})$ with $p_{n_k} \in \mathcal{A}_{n_k}$ such that

$$\lim_{k\to\infty} \frac{M_0(p_{n_k})}{\sqrt{n_k}} = 1\,?$$

• P26. Is it true that $\|Q\|_{0,2\pi]} \le T_{2n}(\sec(s/4))$ for all trigonometric polynomials $Q \in \mathcal{T}_n$ with

$$m(\{t \in [0, 2\pi) : |Q(t)| \le 1\}) \ge 2\pi - s?$$

• P27. Is it true that $\|Q\|_{0,2\pi]} \le T_{2n}(\sec(s/4))$ for all *even* trigonometric polynomials $Q \in \mathcal{T}_n$ with

$$m(\{t \in [0, 2\pi) : |Q(t)| \le 1\}) \ge 2\pi - s?$$

• P28. Is there an absolute constant $c > 0$ such that

$$\|P'\|_{[-1,1]} \le cnm\|P\|_{[-1,1]}$$

for all polynomials $P \in \mathcal{P}_n^c$ with at most $m$ *distinct* zeros (the zeros are allowed to be repeated)?

• P29. (Chowla) What is

$$m(n) := \max_{Q\in\mathcal{B}_n} \min_{t\in\mathbb{R}} Q(t)\,?$$

Is it true that there is an absolute constant $c > 0$ such that $m(n) \le -c\sqrt{n}$?

• P30. What is

$$m^*(n) := \max_{p\in\mathcal{L}_n} \min_{t\in\mathbb{R}} \mathrm{Re}(p)(t)\,?$$

Is it true that there is an absolute constant $c > 0$ such that $m^*(n) \le -c\sqrt{n}$?

• P31. Let $d(p)$ denote the Lorentz degree of a polynomial $p$. Is it true that

$$d(pq) \geq |d(p) - d(q)|$$

for any polynomials $p$ and $q$?

• P32. Let $d(p)$ denote the Lorentz degree of a polynomial $p$. Is it true that

$$d(pq) \geq \min\{|d(p)|, |p(q)|\}?$$

for any polynomials $p$ and $q$?

• P33. Let $P$ be a polynomial of exactly $d$ variables with integral coefficients (the degree of $P$ can be arbitrary). Is it true that the maximum modulus of $P$ on the cube $[-2, 2]^d$ is always at least $cd$ (if not $2d$) with an absolute constant $c > 0$?

• P34. Let $N(p)$ denote the number of zeros of $p$ in the period $[0, 2\pi)$? Is it true that $N(p_n) \to \infty$ whenever $p_n \in \mathcal{B}_n$?