

# MÜNTZ SPACES AND REMEZ INEQUALITIES

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ABSTRACT. Two relatively long standing conjectures concerning Müntz polynomials are resolved. The central tool is a bounded Remez type inequality for non-dense Müntz spaces.

## 1. INTRODUCTION

Müntz's beautiful, classical theorem characterizes sequences  $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$  with

$$(1.1) \quad 0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

for which the Müntz space  $M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$  is dense in  $C[0, 1]$ . Here, and in what follows,  $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$  denotes the collection of finite linear combinations of the functions  $x^{\lambda_0}, x^{\lambda_1}, \dots$  with real coefficients and  $C[A]$  is the space of all real-valued continuous functions on  $A \subset [0, \infty)$  equipped with the uniform norm. Throughout this paper  $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$  denotes a sequence satisfying (1.1). Müntz's Theorem [11, 17, 24, 27] states the following.

**Theorem.**  $M(\Lambda)$  is dense in  $C[0, 1]$  if and only if  $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ .

The original Müntz Theorem proved by Müntz [17] in 1914, by Szász [24] in 1916, and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing  $[0, 1]$  by an interval  $[a, b] \subset [0, \infty)$  in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [12] and Schwartz [22] whose works include the result that if  $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$  then every function belonging to the uniform closure of  $M(\Lambda)$  on  $[a, b]$  can be extended analytically throughout the region  $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}$ .

There are many variations and generalizations of Müntz's Theorem [1, 4, 5, 6, 7, 8, 9, 16, 18, 22, 23, 25, 26]. There are also still many open problems.

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In Section 3 of this paper we show that the interval  $[0, 1]$  in Müntz's Theorem can be replaced by an arbitrary compact set  $A \subset [0, \infty)$  of positive Lebesgue measure. That is, if  $A \subset [0, \infty)$  is a compact set of positive Lebesgue measure, then  $M(\Lambda)$  is dense in  $C[A]$  if and only if  $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ .

If  $A$  contains an interval then this follows from the already mentioned results of Clarkson, Erdős, and Schwartz. However, their results and methods cannot handle the case when, for example,  $A \subset [0, 1]$  is a Cantor type set of positive measure.

In the case that  $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ , analyticity properties of the functions belonging to the uniform closure of  $M(\Lambda)$  on  $A$  are also established.

Speculations about the above extension of Müntz's Theorem are probably as old as Müntz's Theorem itself.

Somorjai [23] and Bak and Newman [2, 19] proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in  $C[0, 1]$ . This surprising result says that while the set  $M(\Lambda)$  of Müntz polynomials may be far from dense, the set  $R(\Lambda)$  of Müntz rationals is always dense in  $C[0, 1]$  no matter what the underlying sequence  $\Lambda$ . In light of this result, Newman, in 1978 [19, p. 50] raises "the very sane, if very prosaic question". Are the functions

$$\prod_{j=1}^k \left( \sum_{i=0}^{n_j} a_{i,j} x^{i^2} \right), \quad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N}$$

dense in  $C[0, 1]$  for some fixed  $k \geq 2$ ? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by  $H_k$ . Since every natural number is the sum of four squares,  $H_4$  contains all the monomials  $x^n$ ,  $n = 0, 1, 2, \dots$ . However,  $H_k$  is not a linear space, so Müntz's Theorem itself cannot be applied. Section 4 of this paper deals with products of Müntz spaces and answers the above question of Newman in the negative. For

$$(1.2) \quad \Lambda_j := \{\lambda_{i,j}\}_{i=0}^{\infty}, \quad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \dots, \quad j = 1, 2, \dots$$

we define the sets

$$M(\Lambda_1, \Lambda_2, \dots, \Lambda_k) := \left\{ p = \prod_{j=1}^k p_j : p_j \in M(\Lambda_j) \right\}.$$

In Section 4 a bounded Remez type inequality is established for  $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  whenever

$$(1.3) \quad \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty, \quad j = 1, 2, \dots, k.$$

This obviously implies that if (1.2) and (1.3) hold and  $A \subset [0, \infty)$  is a compact set of positive Lebesgue measure then  $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  is not dense in  $C[A]$ . In particular,  $H_4$  is not dense in  $C[0, 1]$  which answers Newman's problem negatively. In addition, under the assumptions (1.2) and (1.3), our methods give an "almost characterization" of the uniform closure of  $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  on  $A$  in terms of analyticity properties.

2. BOUNDED REMEZ TYPE INEQUALITY FOR  $M(\Lambda)$ 

Let  $\mathcal{P}_n$  denote the set of all algebraic polynomials of degree at most  $n$  with real coefficients. For a fixed  $s \in (0, 1)$  let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m(\{x \in [0, 1] : |p(x)| \leq 1\}) \geq s\}$$

where  $m(\cdot)$  denotes linear Lebesgue measure. The classical Remez inequality concerns the problem of bounding the uniform norm of a polynomial  $p \in \mathcal{P}_n$  on  $[0, 1]$  given that its modulus is bounded by 1 on a subset of  $[0, 1]$  of Lebesgue measure at least  $s$ . That is, how large can  $\|p\|_{[0,1]}$  (the uniform norm of  $p$  on  $[0, 1]$ ) be if  $p \in \mathcal{P}_n(s)$ ? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials  $\pm T_n(x) := \pm \cos(n \arccos h(x))$ , where  $h$  is a linear function which scales  $[0, s]$  or  $[1 - s, 1]$  onto  $[-1, 1]$ . For various proofs, extensions, and applications see [13, 14, 15, 20, 21].

We announce the following bounded Remez type inequality for  $M(\Lambda)$  whose proof, which is quite difficult, will appear elsewhere.

**Theorem 2.1.** *Suppose  $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ . Let  $s > 0$ . Then there exists a constant  $c$  depending only on  $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$  and  $s$  (and not on  $\varrho$ ,  $A$ , or the “length” of  $p$ ) so that*

$$\|p\|_{[0,\varrho]} \leq c\|p\|_A$$

for every  $p \in M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ , and for every set  $A \subset [\varrho, 1]$  of Lebesgue measure at least  $s$ .

In the above theorem, and throughout the paper,  $\|p\|_A := \sup_{x \in A} |p(x)|$ .

One might note that the existence of such a bounded Remez type inequality for a Müntz space  $M(\Lambda)$  is equivalent to the non-denseness of  $M(\Lambda)$  in  $C[0, 1]$ . We believe that this result should be a basic tool for dealing with problems about Müntz spaces. In this paper we demonstrate the power of Theorem 2.1 by settling two long standing conjectures as fairly straightforward corollaries.

## 3. MÜNTZ'S THEOREM ON COMPACT SETS OF POSITIVE MEASURE

**Theorem 3.1.** *Suppose  $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$  and  $A \subset [0, \infty)$  is a set of positive Lebesgue measure. Then  $M(\Lambda)$  is not dense in  $C[A]$ . Moreover, if the gap condition*

$$(3.1) \quad \inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0$$

holds, then every function  $f \in C[A]$  from the uniform closure of  $M(\Lambda)$  on  $A$  is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \quad x \in A \cap [0, r_A)$$

where  $r_A := \sup\{x \in [0, \infty) : m(A \cap (x, \infty)) > 0\}$  is the essential supremum of  $A$ . If the gap condition (3.1) does not hold, then every function  $f \in C[A]$  from the uniform closure of  $M(\Lambda)$  on  $A$  can still be extended analytically throughout the region  $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$ .

*Proof.* Suppose  $f \in C[A]$  and suppose there is a sequence  $\{p_i\}_{i=1}^\infty \subset M(\Lambda)$  which converges to  $f$  uniformly on  $A$ . Then the sequence  $\{p_i\}_{i=1}^\infty$  is uniformly Cauchy on  $A$ . Therefore, Theorem 2.1 and the definition of  $r_A$  yield that  $\{p_i\}_{i=1}^\infty$  is uniformly Cauchy on every closed subinterval of  $[0, r_A)$ . If the gap condition (3.1) holds then the characterization of the uniform closure of  $M(\Lambda)$  on  $A$  follows from the results of Clarkson and Erdős [12]. If the gap condition (3.1) does not hold, then results of Schwartz [22] yield the theorem.  $\square$

**Theorem 3.2.** *Suppose  $A \subset [0, \infty)$  is a compact set of positive Lebesgue measure. Then  $M(\Lambda)$  is dense in  $C[A]$  if and only if  $\sum_{i=1}^\infty 1/\lambda_i = \infty$ .*

*Proof.* Suppose  $\sum_{i=1}^\infty 1/\lambda_i = \infty$ . Let  $f \in C[A]$ . By Tietze's Extension Theorem there exists an  $\tilde{f} \in C[0, 1]$  so that  $\tilde{f}(x) = f(x)$  for every  $x \in A$ . By Müntz's Theorem there is a sequence  $\{p_i\}_{i=1}^\infty \subset M(\Lambda)$  which converges to  $\tilde{f}$  uniformly on  $[0, 1]$ , hence on  $A$ . This finishes the trivial part of the theorem.

Suppose now that  $\sum_{i=1}^\infty 1/\lambda_i < \infty$ . Then Theorem 3.1 yields that  $M(\Lambda)$  is not dense in  $C[A]$ .  $\square$

#### 4. PRODUCTS OF MÜNTZ SPACES

We prove the following Remez type inequality for  $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ .

**Theorem 4.1.** *Suppose (1.2) and (1.3) hold. Let  $s > 0$ . Then there exists a constant  $c$  depending only on  $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$ , and  $k$  (and not on  $\varrho$  or  $A$ ) so that*

$$\|p\|_{[0, \varrho]} \leq c\|p\|_A$$

for every  $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  and for every set  $A \subset [\varrho, 1]$  of Lebesgue measure at least  $s$ .

*Proof.* Theorem 2.1 implies that there exist constants  $\alpha_j > 0$  depending only on  $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$ , and  $k$  so that

$$m(\{x \in [y, 1] : |p(x)| > \alpha_j^{-1}|p(y)|\}) \geq 1 - y - \frac{s}{2k}$$

for every  $p \in M(\Lambda_j)$  and  $y \in [0, 1 - s]$ . Now let  $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ , that is,  $p = \prod_{j=1}^k p_j$  with  $p_j \in M(\Lambda_j)$ . Then, for every  $y \in [0, 1 - s]$ ,

$$\begin{aligned} & m(\{x \in [y, 1] : |p(x)| > (\alpha_1 \alpha_2 \cdots \alpha_k)^{-1}|p(y)|\}) \\ & \geq m(\cap_{j=1}^k \{x \in [y, 1] : |p_j(x)| > \alpha_j^{-1}|p_j(y)|\}) \\ & \geq 1 - y - k \frac{s}{2k} = 1 - y - \frac{s}{2}. \end{aligned}$$

Hence  $y \in [0, \inf A]$  and  $m(A) \geq s$  imply

$$m(\{x \in A : |p(x)| > (\alpha_1 \alpha_2 \cdots \alpha_k)^{-1} |p(y)|\}) \geq \frac{s}{2} > 0$$

and the theorem follows with  $c = \alpha_1 \alpha_2 \cdots \alpha_k$ .  $\square$

Theorem 4.1 immediately solves Newman's problem [19].

**Corollary 4.2.** *Suppose (1.2) and (1.3) hold and  $A \subset [0, 1]$  is a set of positive Lebesgue measure. Then  $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$  is not dense in  $C[A]$ .*

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