

# MÜNTZ SYSTEMS AND MÜNTZ–LEGENDRE POLYNOMIALS

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ABSTRACT. The Müntz–Legendre polynomials arise by orthogonalizing the Müntz system  $\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$  with respect to the Lebesgue measure on  $[0, 1]$ . In this paper, differential and integral recurrence formulas for the Müntz–Legendre polynomials are obtained. Interlacing and lexicographical properties of their zeros are studied, and the smallest and largest zeros are universally estimated via the zeros of Laguerre polynomials. The uniform convergence of the Christoffel functions is proved equivalent to the nondenseness of the Müntz system, which implies that in this case the Müntz–Legendre polynomials tend to 0 uniformly on closed subintervals of  $[0, 1]$ . Some inequalities for Müntz polynomials are also investigated, in particular, a sharp  $L^2$  Markov inequality is proved.

## §1. INTRODUCTION

Let  $0 \leq \lambda_0 < \lambda_1 < \dots \rightarrow \infty$ . The classical Müntz–Szász Theorem states that the Müntz polynomials of the form  $\sum_{k=0}^n a_k x^{\lambda_k}$  are dense in  $L^2[0, 1]$  if and only if

$$\sum_{k=1}^{\infty} \lambda_k^{-1} = +\infty. \quad (1.1)$$

If the constant function 1 is also in the system, that is,  $\lambda_0 = 0$ , then the denseness of the Müntz polynomials in  $C[0, 1]$  with the uniform norm is also characterized by (1.1). It is our intention to examine various facets of the Müntz system

$$M = \text{Span}\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

and in particular to derive various properties and inequalities of

$$M_n = \text{Span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

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It has been observed [Tas, Mathematical Review 88e:33008] and [MSS], but does not appear to be particularly well known, that the orthogonal “polynomials” (with respect to Lebesgue measure) on  $[0, 1]$  can be explicitly written down. This is the key tool for the analysis we undertake. We prove for example the  $L^2$  Markov inequality

$$\frac{\|xp'(x)\|_2}{\|p\|_2} \leq \frac{1}{\sqrt{2}} \sum_{k=0}^n (1 + 2\lambda_k)$$

for all Müntz polynomials  $p$  in  $M_n$ . Compare this with the  $L^\infty$  result in [New1]

$$\frac{\|xp'(x)\|_\infty}{\|p\|_\infty} \leq 11 \sum_{k=0}^n \lambda_k.$$

Both of these are sharp up to the constants. In order to prove this result and various of its relatives we first derive some explicit formulae and recursions for the sequence of Müntz–Legendre polynomials. Because this orthogonalization is not well known and for completeness we briefly reprove some of the basic formulae which may be found in [Tas, MSS]. This is contained in section 2. Section 3 offers some inequalities for Müntz polynomials, mainly, the above mentioned  $L^2$  Markov inequality. In section 4, we study the interlacing and lexicographical properties of zeros of Müntz–Legendre polynomials and their derivatives. Also in this section, universal estimates of the smallest and largest zeros of Müntz–Legendre polynomials are obtained through the zeros of Laguerre polynomials. Finally in the last section, we look into properties of the Christoffel functions, whose pointwise or uniform convergence on closed subintervals of  $[0, 1]$  is used to characterize the nondenseness of the Müntz system.

Proofs of the Müntz–Szász Theorem can be found in [Che], [FeNe], and [Gol], and various new developments are in [And], [Bor], [BoEr1], [BoEr2], [BoSa], [ClEr], [FeNe], [Lev], [New1], [New2], [New3], [Sch], [Smi], [Som], [Tre], [Zho]. A very special class of Müntz systems, the incomplete polynomials of the form  $x^m p(x)$  where  $p$ 's are the regular polynomials, has been studied intensively (cf. [Lor, SaVa]).

## §2. BASIC PROPERTIES OF MÜNTZ–LEGENDRE POLYNOMIALS

Throughout this paper, we adopt the following definition for  $x^\lambda$ :

$$x^\lambda = e^{\lambda \log x}, \quad x \in (0, 1], \lambda \in \mathbb{C} \tag{2.1}$$

and the value at  $x = 0$  is defined to be the limiting value of  $x^\lambda$  as  $x \rightarrow 0$  from  $(0, 1]$  whenever the limit exists. Given a complex sequence  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ , a finite linear combination of the Müntz system  $\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots\}$  is called a Müntz

*polynomial*, or a  $\Lambda$ -*polynomial*. Denote the set of all such polynomials by  $M(\Lambda)$ , that is,

$$M(\Lambda) = \text{Span}\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots\}. \quad (2.2)$$

Also the collection of linear combinations of the first  $n + 1$  functions is denoted by

$$M_n(\Lambda) = \text{Span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}, \quad (2.3)$$

where the linear span can be over all the reals (this section and §3) or the complexes (§4 and §5), according to context. For the  $L^2$  theory of a Müntz system, we consider

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}, \quad \text{Re}(\lambda_k) > -1/2, \quad \text{and} \quad \lambda_k \neq \lambda_j \ (k \neq j). \quad (2.4)$$

This ensures that every  $\Lambda$ -polynomial is in  $L^2[0, 1]$ . We can then define the orthogonal polynomials with respect to the Lebesgue measure, the Müntz–Legendre polynomials. Although we almost always assume (2.4), the following definition does not require the distinctness of  $\lambda_k$ .

**Definition 2.1.** Let  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  be a complex sequence. We define the  $n$ -th Müntz–Legendre polynomial on  $(0, 1]$  to be [cf. Tas]

$$L(\lambda_0, \lambda_1, \dots, \lambda_n; x) = \frac{1}{2\pi i} \int_{\Gamma} \prod_{k=0}^{n-1} \frac{t + \bar{\lambda}_k + 1}{t - \lambda_k} \frac{x^t dt}{t - \lambda_n} \quad n = 0, 1, 2, \dots, \quad (2.5)$$

where the simple contour  $\Gamma$  surrounds all the zeros of the denominator in the integrand, and  $\bar{\lambda}$  denotes conjugators.

An immediate consequence of the the definition and the residue theorem is

**Corollary 2.2.** Let  $\{\lambda_0, \lambda_1, \lambda_2, \dots\}$  satisfy (2.4). Then for every  $n = 0, 1, 2, \dots$ ,

$$L(\lambda_0, \dots, \lambda_n; x) = \sum_{k=0}^n c_{k,n} x^{\lambda_k}, \quad c_{k,n} = \frac{\prod_{j=0}^{n-1} (\lambda_k + \bar{\lambda}_j + 1)}{\prod_{j=0, j \neq k}^n (\lambda_k - \lambda_j)} \quad (2.6)$$

with  $L(\lambda_0, \dots, \lambda_n; x)$  defined by (2.5).

So,  $L(\lambda_0, \dots, \lambda_n)$  is indeed a Müntz polynomial provided that  $\lambda_0, \lambda_1, \dots, \lambda_n$  are distinct. Its value at  $x = 0$  is defined if for all  $k$  either  $\text{Re}(\lambda_k) > 0$  or  $\lambda_k = 0$ . For example, if  $\lambda_0 = 0$  and  $\text{Re}(\lambda_k) > 0$  ( $1 \leq k \leq n$ ), then  $L(\lambda_0, \dots, \lambda_n; 0) = c_{0,n}$ , and it is 0 if  $\text{Re}(\lambda_0) > 0$  also holds.

*Remark.* From either Definition 2.1 or Corollary 2.2, it is obvious that in  $L(\lambda_0, \dots, \lambda_n)$ , the order of  $\lambda_0, \dots, \lambda_{n-1}$  does not make any difference, as long as  $\lambda_n$  is kept last. For example,  $L(\lambda_0, \lambda_1, \lambda_2) = L(\lambda_1, \lambda_0, \lambda_2)$ , but both are different from  $L(\lambda_0, \lambda_2, \lambda_1)$ . For a fixed (ordered) sequence  $\Lambda$ , we will use  $L_n(\Lambda)$ , or simply  $L_n$  to denote the  $n$ th Müntz–Legendre polynomial  $L(\lambda_0, \dots, \lambda_n)$ , whenever there is no confusion.

In (2.6), repeated indices (for example,  $\lambda_0 = \lambda_1$ ) will cause a problem. But in the original definition,  $\lambda_k = \lambda_j$  is allowed. We can view this also as a limiting process ( $\lambda_k \rightarrow \lambda_j$ ). We state a very special case when all indices are the same, which turns out to be closely related to the Laguerre polynomials. Notice also that the result is actually no longer a Müntz polynomial, with  $\log x$  coming into the picture.

**Corollary 2.3.** *Let  $L(\lambda_0, \dots, \lambda_n; x)$  be defined by (2.5). If  $\lambda_0 = \dots = \lambda_n = \lambda$ , then*

$$L(\lambda_0, \dots, \lambda_n; x) = x^\lambda L_n(-(1 + \lambda + \bar{\lambda}) \log x), \quad (2.7)$$

where  $L_n$  is the  $n$ -th Laguerre polynomial orthonormal with respect to the weight  $e^{-x}$  on  $(0, \infty)$  with  $L_n(0) = 1$ .

*Proof.* Since  $\lambda_k = \lambda$  for  $k = 0, 1, \dots, n$ , by (2.5),

$$L(\lambda_0, \lambda_1, \dots, \lambda_n; x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{x^t (t + \bar{\lambda} + 1)^n}{(t - \lambda)^{n+1}} dt$$

where the contour  $\Gamma$  can be taken to be any circle centered at  $\lambda$ . By the residue theorem,

$$\begin{aligned} L(\lambda_0, \dots, \lambda_n; x) &= \frac{d^n}{n! dt^n} [x^t (t + \bar{\lambda} + 1)^n]_{t=\lambda} \\ &= \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^\lambda (\log x)^k n(n-1) \cdots (k+1) (\lambda + \bar{\lambda} + 1)^k \\ &= x^\lambda \sum_{k=0}^n \frac{1}{k!} \binom{n}{k} (1 + \lambda + \bar{\lambda})^k \log^k x. \end{aligned}$$

These are just the Laguerre polynomials  $\{L_n\}$  which are orthogonal with respect to the weight function  $e^{-x}$  on  $(0, \infty)$  with the normalization  $L_n(0) = 1$  (cf. [Sze, p. 100]), and we obtain (2.7).  $\square$

The name Müntz–Legendre polynomial is justified by the following theorem, where the orthogonality of  $\{L_n\}$  with respect to the Lebesgue measure is proved.

**Theorem 2.4.** *Let  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  satisfy  $\operatorname{Re}(\lambda_k) > -1/2$  for  $k = 0, 1, 2, \dots$ . Assume that  $\{L_n\}_{n=0}^\infty$  are defined by (2.5). Then*

$$\int_0^1 L_n(x) \overline{L_m(x)} = \delta_{n,m} / (1 + \lambda_n + \bar{\lambda}_n) \quad (2.8)$$

holds for every  $m, n = 0, 1, 2, \dots$ .

*Remark.* In the orthogonality (2.8), repeated indices are allowed.

*Proof.* We provide a proof here for the sake of completeness. It suffices to consider  $0 \leq m \leq n$ . Also, we just need to prove (2.8) for distinct indices, since from the definition in (2.5),  $L(\lambda_0, \dots, \lambda_n; x)$  is uniformly continuous in  $\lambda_0, \dots, \lambda_n$  for  $x$  in closed subintervals of  $(0, 1]$ , and the non-distinct case is a limiting argument. Since  $\operatorname{Re}(\lambda_k) > -1/2$ , we can pick a contour  $\Gamma$  in the integral (2.5) such that  $\Gamma$  lies completely to the right of the vertical line  $\operatorname{Re}(t) = -1/2$ , and  $\Gamma$  surrounds all zeros

of the denominator. When  $t \in \Gamma$ , we have  $\operatorname{Re}(t + \bar{\lambda}_m) > -1$ , and  $\int_0^1 x^{t+\bar{\lambda}_m} dx = 1/(1+t+\bar{\lambda}_m)$ , for every  $m \geq 0$ . Hence,

$$\int_0^1 L_n(x) \overline{x^{\lambda_m}} dx = \int_{\Gamma} \prod_{k=0}^{n-1} \frac{t + \bar{\lambda}_k + 1}{t - \lambda_k} \frac{dt}{(t - \lambda_n)(t + \bar{\lambda}_m + 1)}.$$

Notice that for  $m < n$ , the new term  $t + \bar{\lambda}_m + 1$  in the denominator can be cancelled, and for  $m = n$  the new pole  $-(\bar{\lambda}_n + 1)$  is out side  $\Gamma$ , because  $\operatorname{Re}(-\bar{\lambda}_n - 1) < -1/2$ . Changing the contour from  $\Gamma$  to  $|t| = R$  with  $R > \max\{|\lambda_0| + 1, \dots, |\lambda_n| + 1\}$ , we have for  $0 \leq m \leq n$  that

$$\begin{aligned} \int_0^1 L_n(x) \overline{x^{\lambda_m}} dx &= \frac{1}{2\pi i} \int_{|t|=R} \prod_{k=0}^{n-1} \frac{t + \bar{\lambda}_k + 1}{t - \lambda_k} \frac{dt}{(t - \lambda_n)(t + \bar{\lambda}_n + 1)} \\ &\quad - \frac{\delta_{m,n}}{-\bar{\lambda}_n - 1 - \lambda_n} \prod_{k=0}^{n-1} \frac{-\bar{\lambda}_n + \lambda_k}{-\bar{\lambda}_n - 1 - \lambda_k}. \end{aligned}$$

Letting  $R \rightarrow \infty$ , we see that the integral on the right-hand side is actually 0, which gives

$$\int_0^1 L_n(x) \overline{x^{\lambda_m}} dx = \frac{\delta_{n,m}}{\lambda_n + \lambda_n + 1} \prod_{k=0}^{n-1} \frac{\bar{\lambda}_n - \lambda_k}{\lambda_n + \lambda_k + 1}.$$

Now with (2.6), we have for  $0 \leq m \leq n$  that

$$\begin{aligned} \int_0^1 L_n(x) \overline{L_m(x)} dx &= \int_0^1 L_n(x) \sum_{k=0}^m \overline{c_{k,m} x^{\lambda_k}} dx \\ &= \overline{c_{m,m}} \int_0^1 L_n(x) x^{\bar{\lambda}_m} dx = \delta_{m,n}/(\lambda_n + \bar{\lambda}_n + 1), \end{aligned}$$

where the last step comes from the formula for  $c_{k,n}$  in (2.6)  $\square$

An alternative and probably easier proof of orthogonality follows from (2.10) below, integration by parts and induction. Later we will see that  $L_n(x) = 1$  when  $x = 1$ . This can be viewed as the normalization for  $L_n$ . Clearly, if we let

$$L_n^* = (1 + \lambda_n + \bar{\lambda}_n)^{1/2} L_n, \quad (2.9)$$

then we get an orthonormal system, that is,

$$\int_0^1 L_n^*(x) \overline{L_m^*(x)} dx = \delta_{m,n}, \quad m, n = 0, 1, 2, \dots$$

There is also a Rodrigues formula for the Müntz–Legendre polynomials [MSS]. Let  $p_n(x) = \sum_{k=0}^n x^{\lambda_k} / \prod_{j=0, j \neq k}^n (\lambda_k - \lambda_j)$ , then

$$L_n(x) = D_{\lambda_0} \cdots D_{\lambda_{n-1}} p_n(x)$$

where the differential operator  $D_{\lambda}$ 's are defined by  $D_{\lambda} f = x^{-\bar{\lambda}} \frac{d}{dx} x^{1+\bar{\lambda}} f$ . Notice also that  $p_n$  and its first  $n - 1$  derivatives vanish at  $x = 1$ .

Now we state the differential recurrence formulas for  $\{L_n\}$

**Theorem 2.5.** *Assume that  $\Lambda$  is a complex sequence satisfying  $\operatorname{Re}(\lambda_k) > -1/2$  for all  $k$ . Then*

$$xL'_n(x) - xL'_{n-1}(x) = \lambda_n L_n(x) + (1 + \bar{\lambda}_{n-1})L_{n-1}(x) \quad n = 1, 2, 3, \dots, \quad (2.10)$$

where the  $\{L_k\}$  are the associated Müntz–Legendre polynomials defined by (2.5).

*Proof.* From (2.5), we get

$$\frac{d}{dx} (x^{-\lambda_n} L_n(x)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\prod_{k=0}^{n-2} (t + \bar{\lambda}_k + 1)}{\prod_{k=0}^{n-1} (t - \lambda_k)} (t + \bar{\lambda}_{n-1} + 1) x^{t-\lambda_n-1} dt.$$

Multiplying by  $x^{\lambda_n + \bar{\lambda}_{n-1} + 1}$  on both sides in the above, we obtain

$$x^{\lambda_n + \bar{\lambda}_{n-1} + 1} (x^{-\lambda_n} L_n(x))' = \frac{1}{2\pi i} \int_{\Gamma} \frac{\prod_{k=0}^{n-2} (t + \bar{\lambda}_k + 1)}{\prod_{k=0}^{n-1} (t - \lambda_k)} (t + \bar{\lambda}_{n-1} + 1) x^{t + \bar{\lambda}_{n-1}} dt,$$

and again by the definition of  $L_{n-1}$  (cf. (2.5)),

$$x^{\lambda_n + \bar{\lambda}_{n-1} + 1} (x^{-\lambda_n} L_n(x))' = \left( x^{\bar{\lambda}_{n-1} + 1} L_{n-1}(x) \right)'$$

Simplifying the above by the product rule and dividing both sides by  $x^{\bar{\lambda}_{n-1}}$  gives (2.10).  $\square$

**Corollary 2.6.** *Let a complex sequence  $\Lambda$  satisfy (2.4), and let the associated Müntz–Legendre polynomials  $L_n$  and the orthonormal Müntz polynomials  $\{L_n^*\}$  be defined by (2.5) and (2.9) respectively, then*

$$xL'_n(x) = \lambda_n L_n(x) + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1) L_k(x), \quad (2.11)$$

$$xL_n^{*'}(x) = \lambda_n L_n^*(x) + \sqrt{\lambda_n + \bar{\lambda}_n + 1} \sum_{k=0}^{n-1} \sqrt{\lambda_k + \bar{\lambda}_k + 1} L_k^*(x), \quad (2.12)$$

and

$$xL_n''(x) = (\lambda_n - 1)L'_n(x) + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1)L'_k(x) \quad (2.13)$$

for every  $x \in (0, 1]$  and every  $n = 0, 1, 2, \dots$ .

*Proof.* The first equality (2.11) follows from Theorem 2.4 by writing  $xL'_n(x) - xL'_0(x)$  as a telescoping sum. From (2.11) and the relation  $L_k^* = (\lambda_k + \bar{\lambda}_k + 1)^{1/2} L_k$  (cf. (2.9)), we get (2.12), and differentiating (2.11) gives (2.13).  $\square$

The values and derivative values of the Müntz–Legendre polynomials at 1 can all be calculated. They are useful in locating the zeros of Müntz–Legendre polynomials (cf. [84]).

**Corollary 2.7.** *Let  $L_n$  be the  $n$ th Müntz-Legendre polynomial defined by (2.5) (or by (2.6) from  $\Lambda$  satisfying (2.4)), then*

$$L_n(1) = 1, \quad L'_n(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1), \quad n = 0, 1, 2, \dots, \quad (2.14)$$

and

$$L''_n(1) = (\lambda_n - 1)L'_n(1) + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1)L'_k(1) \quad n = 0, 1, 2, \dots \quad (2.15)$$

*Proof.* It suffices to show that  $L_n(1) = 1$ , for the rest follows from Corollary 2.6. Notice that from (2.5),

$$L_n(1) = \frac{1}{2\pi i} \int_{\Gamma} \prod_{k=0}^{n-1} \frac{t + \bar{\lambda}_k + 1}{t - \lambda_n} \frac{dt}{t - \lambda_n}.$$

Since  $\Gamma$  surrounds all zeros of the denominator, and the degree of the denominator is 1 higher than the numerator, let  $\Gamma$  be the circle  $|t| = R$  and let  $R \rightarrow \infty$ . From this we get  $L_n(1) = 1$ .  $\square$

The recurrence formula can also be expressed in integral form.

**Corollary 2.8.** *Let a complex sequence  $\Lambda$  be given as (2.4), and Let  $\{L_k\}$  be the Müntz-Legendre polynomials defined by (2.5). Then,*

$$L_n(x) = L_{n-1}(x) - (\lambda_n + \bar{\lambda}_{n-1} + 1)x^{\lambda_n} \int_x^1 x^{-\lambda_n-1} L_{n-1}(t) dt \quad x \in (0, 1] \quad (2.16)$$

*Proof.* Rewrite the recurrence formula (2.10) as

$$xL_n(x) - \lambda_n L_n(x) = xL'_{n-1}(x) + (1 + \bar{\lambda}_{n-1})L_{n-1}(x),$$

and multiply both sides by  $x^{-\lambda_n-1}$  to get

$$(x^{-\lambda_n} L_n(x))' = x^{-\lambda_n} L'_{n-1}(x) + (1 + \bar{\lambda}_{n-1})x^{-\lambda_n-1} L_{n-1}(x).$$

On taking the definite integral of the above on  $[x, 1]$ , and using the fact that  $L_k(1) = 1$  for all  $k \geq 0$ , we have

$$\begin{aligned} 1 - x^{-\lambda_n} L_n(x) &= 1 - x^{-\lambda_n} L_{n-1}(x) - \int_x^1 (t^{-\lambda_n})' L_{n-1}(t) dt \\ &\quad + (\bar{\lambda}_{n-1} + 1) \int_x^1 t^{-\lambda_n-1} L_{n-1}(t) dt, \end{aligned}$$

which clearly implies (2.16).  $\square$

Another observation is that if  $0 \leq \lambda_n \rightarrow \infty$  very fast, then  $x = 1$  is the unique maximal point of the Müntz-Legendre polynomial on  $[0, 1]$

**Corollary 2.9.** *If  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  is a nonnegative sequence such that*

$$\lambda_n \geq \sum_{k=0}^{n-1} (1 + 2\lambda_k), \quad n = 1, 2, 3, \dots, \quad (2.17)$$

then

$$|L_n(x)| < L_n(1) = 1, \quad x \in [0, 1), \quad n = 2, 3, 4, \dots \quad (2.18)$$

*Remark.* If  $\lambda_k = \rho^k$ , then (2.17) holds if and only if  $\rho \geq 2 + \sqrt{3}$ .

*Proof.* We assume  $\lambda_0 = 0$ . (The proof for  $\lambda_0 > 0$  is essentially the same.) In this case,  $L_0(x) \equiv 1$ , and (2.18) fails for  $n = 0$ . From (2.17),  $\lambda_1 \geq 1$ , and  $\lambda_k \geq 2 + \lambda_{k-1}$  for  $k \geq 2$ . By (2.6),

$$|L_n(0)| = |c_{0,n}| = \frac{\prod_{j=0}^{n-1} |1 + \lambda_j|}{\prod_{j=1}^n |\lambda_j|} = \prod_{j=1}^n \frac{1 + \lambda_{j-1}}{\lambda_j}.$$

Hence,  $|L_1(0)| \leq 1$ , and  $L_n(0) < 1$  for  $n \geq 2$ . Now we use induction to show that  $|L_n(x)| < 1$  on  $(0, 1)$  for every  $n \geq 1$ . Indeed, for  $n = 1$ , because  $|L_1(0)| \leq 1 = L_1(1)$ , and  $L_1 = c_{0,1} + c_{1,1}x^{\lambda_1}$  is monotone on  $[0, 1]$ , we have  $|L_1(x)| < 1$  on  $(0, 1)$ . Assume that  $n \geq 2$ , and that  $|L_k(x)| < 1$  for  $1 \leq k \leq n-1$ . Let  $x$  be a local maximal point of  $|L_n|$  in  $(0, 1)$ , then  $L'_n(x) = 0$ . By Corollary 2.6, we have

$$\lambda_n L_n(x) + \sum_{k=0}^{n-1} (1 + 2\lambda_k) L_k(x) = 0.$$

Therefore,

$$|L_n(x)| = \frac{1}{\lambda_n} \left| \sum_{k=0}^{n-1} (1 + 2\lambda_k) L_k(x) \right| < \sum_{k=0}^{n-1} (1 + 2\lambda_k) / \lambda_n \leq 1. \quad \square$$

We finish this section by introducing the reproducing kernels. They are similar to the Dirichlet kernels in the trigonometric theory, or to the reproducing kernels for orthogonal polynomials (cf. [Sze, p. 40]).

**Corollary 2.10.** *Let  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  be as in (2.4), and let  $\{L_n\}$  be defined by (2.5). Then for every Müntz polynomial  $p(x) = \sum_{k=0}^n a_k x^{\lambda_k}$  in  $M_n(\Lambda)$ , we have*

$$p(x) = \int_0^1 K_n(x, t) p(t) dt \quad (2.19)$$

where

$$K_n(x, t) = \sum_{k=0}^n L_k^*(x) \overline{L_k^*(t)} \quad (2.20)$$

is the  $n$ th reproducing kernel.

*Proof.* This is a well-known fact for orthogonal series. By orthonormality of  $L_n^*$ , we have

$$\int_0^1 K_n(x, t) L_k^*(t) dt = L_k^*(x), \quad 0 \leq k \leq n.$$

Since  $\{L_0^*, \dots, L_n^*\}$  is a base of  $M_n(\Lambda)$ , the above is equivalent to (2.19).  $\square$

Later in §3 and §5, we will see the importance of  $K_n$  in polynomial inequalities, and in the denseness of Müntz systems.

### §3. INEQUALITIES FOR MÜNTZ SYSTEMS

Let  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  satisfy (2.4) and the collections of  $M(\Lambda)$  and  $M_n(\Lambda)$  be defined by (2.2) and (2.3). With the help of Müntz–Legendre polynomials, we establish some inequalities for  $\Lambda$ -polynomials.

First, we record an estimate of a Müntz polynomial  $p$  and its derivative at a point  $y \in (0, 1]$  in terms of its  $L^2$  norm ( $\|p\|_2 = (\int_0^1 |p(t)|^2 dt)^{1/2}$ ).

**Theorem 3.1.** *Suppose that  $\Lambda$  satisfies (2.4) and that  $\{L_k^*\}$  are the orthonormal Müntz–Legendre polynomials. Then for any  $\Lambda$ -polynomial  $p \in M_n(\Lambda)$  and any  $v = 0, 1, 2, \dots$*

$$|p^{(\nu)}(y)| \leq \left[ \sum_{k=0}^n |L_k^{*(\nu)}(y)|^2 \right]^{1/2} \|p\|_2 \quad y \in (0, 1], \quad (3.1)$$

where the equality holds if and only if  $p(x) = \text{const} \sum_{k=0}^n \overline{L_k^{*(\nu)}(y)} L_k^*(x)$ .

*Remark.* An equivalent expression of the above is

$$\left[ \sum_{k=0}^n |L_k^{*(\nu)}(y)|^2 \right]^{1/2} = \max\{ |p^{(\nu)}(y)| : p \in M_n(\Lambda), \|p\|_2 = 1 \}, \quad (3.2)$$

which, by letting  $n \rightarrow \infty$ , leads to

$$\left[ \sum_{k=0}^{\infty} |L_k^{*(\nu)}(y)|^2 \right]^{1/2} = \sup\{ |p^{(\nu)}(y)| : p \in M(\Lambda), \|p\|_2 = 1 \} \quad (3.3)$$

which may be finite or infinite. The above will be studied further in §5.

*Proof.* This is also a well-known consequence of the orthogonality of  $\{L_k^*\}$  (cf. [Sze, p. 39]). A simple proof is to use the reproducing kernel of Corollary 2.10 to get

$$p^{(\nu)}(y) = \int_0^1 \frac{\partial^\nu}{\partial y^\nu} K_n(y, t) p(t) dt, \quad p \in M_n(\Lambda)$$

and then apply the Cauchy–Schwartz inequality.  $\square$

We now focus on one of the principal results, the  $L^2$  Markov inequalities for Müntz polynomials, whose  $L^\infty$  version is in [New1]

**Theorem 3.2.** *Assume that  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  is given as in (2.4). Then,*

$$\sup_{p \in M_n(\Lambda)} \frac{\|xp'(x)\|_2}{\|p\|_2} \leq \left[ \sum_{k=0}^n |\lambda_k|^2 + \sum_{k=0}^n (1 + 2\operatorname{Re}\lambda_k) \sum_{j=k+1}^n (1 + 2\operatorname{Re}\lambda_j) \right]^{1/2} \quad (3.4)$$

*If in addition,  $\Lambda$  consists of nonnegative reals, then*

$$\frac{1}{2\sqrt{30}} \sum_{k=0}^n \lambda_k \leq \sup_{p \in M_n(\Lambda)} \frac{\|xp'(x)\|_2}{\|p\|_2} \leq \frac{1}{\sqrt{2}} \sum_{k=0}^n (1 + 2\lambda_k) \quad (3.5)$$

*where  $n$  is an arbitrary nonnegative integer.*

*Remark.* It is easy to see that the imaginary parts of  $\lambda_k$ 's does not affect the Markov factor as much as their real parts. For example, if  $\lambda_k = ik$ , then the Markov bound on the right side of (3.4) is  $[\sum_{k=0}^n (k^2 + n - k)]^{1/2} = O(n^{3/2})$ , while  $\lambda_k = k$  gives  $O(n^2)$ .

*Proof.* Let  $p \in M_n(\Lambda)$  be arbitrary, and  $\|p\|_2 = 1$ . Then  $p(x) = \sum_{k=0}^n a_k L_k^*(x)$ , and  $\|p\|_2^2 = \sum_{k=0}^n |a_k|^2 = 1$ . Thus,

$$xp'(x) = \sum_{k=0}^n a_k x L_k^{*'}(x).$$

If we use the recurrence formula (2.12) for the terms  $xL_k^{*'}$  in the above and rearrange the sum, we get

$$xp'(x) = \sum_{k=0}^n \left[ a_j \lambda_j + \sqrt{1 + \lambda_j + \bar{\lambda}_j} \sum_{k=j+1}^n a_k \sqrt{1 + \lambda_k + \bar{\lambda}_k} \right] L_k^*(x).$$

Hence,  $\int_0^1 |xp'(x)|^2 dx = \sum_{k=0}^n |a_j \lambda_j + \sum_{k=j+1}^n a_k \sqrt{1 + \lambda_k + \bar{\lambda}_k}|^2$ . We apply the Cauchy–Schwartz inequality for each term in the sum, and, on noting that  $\sum_{k=0}^n |a_k|^2 = 1$ , we get

$$\int_0^1 |xp'(x)|^2 dx \leq \sum_{j=0}^n \left[ |\lambda_j|^2 + (1 + \lambda_j + \bar{\lambda}_j) \sum_{k=j+1}^n (1 + \lambda_k + \bar{\lambda}_k) \right] \leq \frac{1}{2} \left[ \sum_{j=0}^n (1 + 2|\lambda_j|) \right]^2.$$

The above proves (3.4) and the right half of (3.5). To prove the sharpness for the case  $\lambda_k \geq 0$  ( $k \geq 0$ ), we need to find a nontrivial  $\Lambda$ -polynomial  $p$  in  $M_n(\Lambda)$ , such that

$$\|xp'\|_2^2 \geq C \left( \sum_{k=0}^n \lambda_k \right)^2 \|p\|_2^2. \quad (3.6)$$

Lemma 3.1 suggests that a possible candidate is  $\sum_{k=0}^n \overline{L_k^{*'}(1)} L_k^*(x)$ , and indeed this works. But a slight alternation makes the estimation easier. We consider

$$p(x) = \sum_{k=0}^n \sqrt{\lambda_k} \left( \sum_{j=0}^k \lambda_j \right) L_k^*(x).$$

By orthonormality of  $L_k^*$ , we have

$$\int_0^1 |p(x)|^2 dx = \sum_{k=0}^n \lambda_k \left( \sum_{j=0}^k \lambda_j \right)^2 \leq \left( \sum_{j=0}^n \lambda_k \right)^3. \quad (3.7)$$

Now

$$xp'(x) = \sum_{k=0}^n \sqrt{\lambda_k} \left( \sum_{j=0}^k \lambda_j \right) x L_k^{*'}(x) = \sum_{s=0}^n b_s L_s^*(x),$$

where by the recurrence formula (2.12)

$$b_s = \lambda_s \sqrt{\lambda_s} \sum_{j=0}^s \lambda_j + \sqrt{1+2\lambda_s} \sum_{k=s+1}^n \sqrt{\lambda_k(1+2\lambda_k)} \sum_{j=0}^k \lambda_j \geq \sqrt{\lambda_s} \sum_{k=s}^n \lambda_k \sum_{j=0}^k \lambda_j \geq 0.$$

Hence,

$$\begin{aligned} \int_0^1 |xp'(x)|^2 dx &= \sum_{s=0}^n |b_s|^2 \geq \sum_{s=0}^n \lambda_s \left( \sum_{k=s}^n \lambda_k \sum_{j=0}^k \lambda_j \right)^2 \\ &= \sum_{\substack{0 \leq s \leq n \\ s \leq k, k' \leq n}} \sum_{\substack{0 \leq j \leq k \\ 0 \leq j' \leq k'}} \lambda_s \lambda_k \lambda_j \lambda_{k'} \lambda_{j'} \geq \sum_{0 \leq s \leq j \leq j' \leq k \leq k' \leq n} \lambda_s \lambda_k \lambda_j \lambda_{k'} \lambda_{j'} \geq \frac{1}{5!} \left( \sum_{k=0}^n \lambda_k \right)^5. \end{aligned}$$

This together with (3.7) proves (3.6), and hence the left half of (3.5).  $\square$

#### §4. ZEROS OF MÜNTZ-LEGENDRE POLYNOMIALS

In [PiZi], the interlacing properties of zeros of the error functions of best  $L^p$  ( $0 \leq p \leq \infty$ ) is thoroughly studied. Many of those results can be applied to Müntz-Legendre polynomials. Indeed,  $L_n(x)/c_{n,n} = x^{\lambda_n} + \dots$  (cf. (2.6)) is the error function of the best  $L^2$  approximation to  $x^{\lambda_n}$  by Müntz polynomials in  $M_{n-1}(\Lambda)$ , that is,

$$\min_{p \in M_{n-1}(\Lambda)} \int_0^1 |x^{\lambda_n} - p(x)|^2 dx = \frac{1}{(1 + 2\operatorname{Re}\lambda_n) |c_{n,n}|^2},$$

which is again a consequence of orthogonality (cf. [Sze, Theorem 3.1.2]). In this section we consider the interlacing and lexicographical properties of Müntz-Legendre

polynomials and their derivatives. Now it is natural to assume that  $\lambda_k$ 's are real and that

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\} \subset (-1/2, \infty), \quad \lambda_k \neq \lambda_j \ (k \neq j). \quad (4.1)$$

Since for every  $n \geq 0$  the Müntz system  $\{x^{\lambda_k}\}_{k=0}^n$  is a Chebyshev system, that is, every nonzero polynomial  $p(x) = \sum_{k=0}^n a_k x^{\lambda_k}$  has at most  $n$  zeros in  $(0,1]$  (with the zeros in  $(0,1)$  that do not change signs counted twice). (That is denoted by  $Z(p) \leq n$ .) The interlacing properties are based on the following Lemma from [PiZi].

**Lemma 4.1.** *[PiZi, proposition 3.2.] Let  $\Phi, \Psi \in C(0,1]$ . If for every real  $\alpha$  and  $\beta$  with  $\alpha^2 + \beta^2 > 0$ , the number of sign changes  $S^-(\alpha\Phi + \beta\Psi)$  and the number of zeros  $Z(\alpha\Phi + \beta\Psi)$  satisfy*

$$n \leq S^-(\alpha\Phi + \beta\Psi) \leq Z(\alpha\Phi + \beta\Psi) \leq n + 1,$$

then the zeros of  $\Phi$  and  $\Psi$  strictly interlace.

Let  $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$  be distinct and bigger than  $-1/2$ . Now the associated Müntz–Legendre polynomial takes the form (cf. (2.6))

$$L(\lambda_0, \dots, \lambda_n; x) = \sum_{k=0}^n c_{k,n} x^{\lambda_k}, \quad c_{k,n} = \frac{\prod_{j=0}^{n-1} (1 + \lambda_j + \lambda_k)}{\prod_{j=0, j \neq k}^n (\lambda_k - \lambda_j)}. \quad (4.2)$$

Either as a simple application of Lemma 4.1 or a direct citation of [PiZi, Theorem 1.1 and Corollary 2], we have

**Corollary 4.2.** *If  $\Lambda$  is given as in (4.1), then, for every  $n = 0, 1, 2, \dots$ ,*

- (1)  $L(\lambda_0, \dots, \lambda_n)$  has exactly  $n$  zeros in  $(0,1)$ ;
- (2) The zeros of  $L(\lambda_0, \dots, \lambda_{n-1})$  and  $L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  strictly interlace.

Now we show what happens to the zeros when the last index is changed. Roughly speaking, if the last index becomes bigger, the zeros interlacingly shift to the right.

**Corollary 4.3.** *Let  $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$  satisfy (4.1). Assume  $-1/2 < \lambda_n^* \neq \lambda_k$  for  $k = 0, 1, \dots, n$ . Then*

- (1) the zeros of  $L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  and  $L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n^*)$  strictly interlace;
- (2) if  $\lambda_n < \lambda_n^*$ , then the zeros move strictly to the right, that is,

$$x_1 < x_1^*, \quad \dots, \quad x_n < x_n^*, \quad (4.3)$$

where  $x_1^* < \dots < x_n^*$  and  $x_1 < \dots < x_n$  are the zeros of  $L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n^*)$

and  $L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  in  $(0,1)$ , respectively.

*Proof.* First we prove the interlacing property of the zeros of  $\Phi = L(\lambda_0, \dots, \lambda_n)$ , and  $\Psi = L(\lambda_0, \dots, \lambda_n^*)$ . By orthogonality in (2.8), we have for every real  $\alpha$  and  $\beta$  that

$$\int_0^1 (\alpha\Phi + \beta\Psi)x^{\lambda_j} dx = 0, \quad j = 0, \dots, n-1. \quad (4.4)$$

We claim that  $S^-(\alpha\Phi + \beta\Psi) \geq n$  when  $\alpha^2 + \beta^2 > 0$ . Indeed, if the number of sign changes of  $\alpha\Phi + \beta\Psi$  in  $(0, 1)$  is not bigger than  $n-1$ , and we can choose a nontrivial Müntz polynomial  $p(x) = \sum_{k=0}^{n-1} a_k x^{\lambda_k}$  such that it changes sign exactly when  $\alpha\Phi + \beta\Psi$  does. Accordingly, we may assume that  $(\alpha\Phi + \beta\Psi)p \geq 0$  on  $(0, 1]$ . But by (4.4),

$$\int_0^1 (\alpha\Phi(x) + \beta\Psi(x))p(x) dx = 0$$

which can happen only when  $(\alpha\Phi + \beta\Psi)p \equiv 0$  on  $(0, 1]$ , a contradiction. Clearly  $Z(\alpha\Phi + \beta\Psi) \leq n+1$  because  $\{x^{\lambda_0}, \dots, x^{\lambda_n}, x^{\lambda_n^*}\}$  is a Chebyshev system. Hence by Lemma 4.1, the zeros of  $\Phi = L(\lambda_0, \dots, \lambda_n)$  and  $\Psi = L(\lambda_0, \dots, \lambda_n^*)$  strictly interlace. Secondly, we show as  $\lambda_n$  increases to  $\lambda_n^*$ , the zeros move strictly to the right. Since the zeros strictly interlace, we see that they move to the right if  $\Psi'(1)/\Psi(1) > \Phi'(1)/\Phi(1)$ . This is indeed the case, since by Corollary 2.7 ,

$$\frac{\Psi'(1)}{\Psi(1)} - \frac{\Phi'(1)}{\Phi(1)} = \lambda_n^* - \lambda_n,$$

which is positive because  $\lambda_n^* > \lambda_n$ .  $\square$

We already know that  $L(\lambda_0, \dots, \lambda_n)$  does not depend on the order of appearance of  $\lambda_0, \dots, \lambda_{n-1}$  if the last one is kept last. The next statement tells us what happens if the last index  $\lambda_n$  is swapped with a previous one  $\lambda_k$  where  $k < n$ , and the other indices are the same. Again, the zeros interlacingly shift to the right if  $\lambda_n > \lambda_k$ .

**Corollary 4.4.** *Let  $\lambda_0, \dots, \lambda_k, \dots, \lambda_n > -1/2$  be distinct. Then,*

- (1) *the zeros of  $L(\lambda_0, \dots, \lambda_k, \dots, \lambda_n)$  and  $L(\lambda_0, \dots, \lambda_n, \dots, \lambda_k)$  strictly interlace;*
- (2) *if  $\lambda_n > \lambda_k$ , then the zeros move strictly to the right, that is,*

$$x_1 < x_1^*, \quad \dots, \quad x_n < x_n^*, \quad (4.5)$$

*where  $x_1^* < \dots < x_n^*$  and  $x_1 < \dots < x_n$  are the zeros of  $L(\lambda_0, \dots, \lambda_n, \dots, \lambda_k)$  and  $L(\lambda_0, \dots, \lambda_k, \dots, \lambda_n)$  in  $(0, 1)$ , respectively.*

*Proof.* The proof is almost identical to that of Corollary 4.3. Let  $\Phi = L(\lambda_0, \dots, \lambda_k, \dots, \lambda_n)$ , and  $\Psi = L(\lambda_0, \dots, \lambda_n, \dots, \lambda_k)$ . Then  $n-1 \leq S^-(\alpha\Phi + \beta\Psi) \leq Z(\alpha\Phi + \beta\Psi) \leq n$ , and Lemma 4.1 proves the interlacing property of the zeros. Also by Corollary 2.7

$$\frac{\Psi'(1)}{\Psi(1)} - \frac{\Phi'(1)}{\Phi(1)} = \lambda_n - \lambda_k,$$

and we conclude that the zeros move strictly to the right if the last index  $\lambda_n$  is swapped with a smaller previous index  $\lambda_k$ .  $\square$

Applying Corollary 4.3 and 4.4 repeatedly, we have

**Theorem 4.5.** *Let  $\{\lambda_0, \dots, \lambda_n\}$  and  $\{\mu_0, \dots, \mu_n\}$  satisfy (4.1). Then*

- (1) *if  $\max\{\lambda_0, \dots, \lambda_n\} \leq \min\{\mu_0, \dots, \mu_n\}$ , then the zeros of  $L(\mu_0, \dots, \mu_n)$  lie to the right of the zeros of  $L(\lambda_0, \dots, \lambda_n)$ ;*
- (2) *if  $\lambda_0 < \dots < \lambda_n$  and  $\mu_0 < \dots < \mu_n$ , and  $\mu_0 \geq \lambda_0, \dots, \mu_n \geq \lambda_n$ , then the zeros of  $L(\mu_n, \dots, \mu_0)$  (note the reversal of the  $\mu_k$ ) lie to the right of the zeros of  $L(\lambda_0, \dots, \lambda_n)$ . In particular, if  $\lambda_0 < \dots < \lambda_n$ , then the zeros of  $L(\lambda_n, \dots, \lambda_0)$  lie to the right of  $L(\lambda_0, \dots, \lambda_n)$ .*

*Proof.* (i.) The zeros of  $L(\mu_0, \dots, \mu_{n-1}, \mu_n)$  lie to the right of the zeros of  $L(\mu_0, \dots, \mu_{n-1}, \lambda_0)$  by Corollary 4.3 because  $\mu_n \geq \lambda_0$  (strictly to the right, if  $\mu_n > \lambda_0$ ), and the zeros of  $L(\mu_0, \dots, \mu_{n-1}, \lambda_0)$  lie to the right of  $L(\mu_0, \dots, \mu_{n-2}, \lambda_0, \mu_{n-1})$ , because of  $\mu_{n-1} \geq \lambda_0$  and Corollary 4.4. Hence the zeros of  $L(\mu_0, \dots, \mu_n)$  lie to the right of  $L(\lambda_0, \mu_0, \dots, \mu_{n-1})$ , because keeping the last index last, and permuting the previous indices does not affect the Müntz–Legendre polynomial. Now in the similar fashion, we replace  $\mu_{n-1}$  by  $\lambda_1$ , and swap  $\lambda_1$  with  $\mu_{n-2}$ , we get that the zeros of  $L(\lambda_0, \mu_0, \dots, \mu_{n-1})$  lie to the right of  $L(\lambda_0, \lambda_1, \mu_0, \dots, \mu_{n-2})$ . Repeating the process, until all  $\mu$ 's are replaced, and the first part follows.

(ii.) This part is very much the same as (i). Corollary 4.3 implies that the zeros of  $L(\mu_n, \dots, \mu_1, \mu_0)$  lie to the right of those of  $L(\mu_n, \dots, \mu_1, \lambda_0)$  because  $\mu_0 > \lambda_0$ . Whereas Corollary 4.4 implies that the zeros of  $L(\mu_n, \dots, \mu_1, \lambda_0)$  lie strictly to the right of  $L(\mu_n, \dots, \mu_2, \lambda_0, \mu_1)$  because  $\mu_1 > \lambda_0$ . Hence the zeros of  $L(\mu_n, \dots, \mu_0)$  lie to the right of  $L(\lambda_0, \mu_n, \dots, \mu_1)$ . Similarly, the zeros of  $L(\lambda_0, \mu_n, \dots, \mu_1)$  lie to the right of those of  $L(\lambda_0, \lambda_1, \mu_n, \dots, \mu_2)$ , and we can keep doing this until all  $\mu$ 's are replaced by  $\lambda$ 's.  $\square$

The interlacing and lexicographical properties of zeros also hold for the derivatives of Müntz–Legendre polynomials. Now we will have a restriction on the index set  $\Lambda$ :  $\lambda_0 = 0$ , which is to ensure that the derivative has one less zero.

**Theorem 4.6.** *Let  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  satisfy (4.1), and  $\lambda_0 = 0$ . Then for every  $n \geq 0$ ,*

- (1) *the zeros of  $L'(\lambda_0, \dots, \lambda_{n-1})$  and  $L'(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  strictly interlace;*
- (2) *if  $\lambda_n^* \neq \lambda_k$ ,  $k = 0, 1, \dots, n$ , then the zeros of  $L'(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  strictly interlace those of  $L'(\lambda_0, \dots, \lambda_{n-1}, \lambda_n^*)$  for  $n \geq 1$ . If  $\lambda_k \geq 0$ ,  $k = 0, 1, \dots, n$ , and  $\lambda_n^* > \lambda_n$ , then the zeros of  $L'(\lambda_0, \dots, \lambda_{n-1}, \lambda_n^*)$  lie strictly to the right of those of  $L'(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$ ;*
- (3) *for every  $k = 0, 1, \dots, n-1$ , the zeros of  $L'(\lambda_0, \dots, \lambda_k, \dots, \lambda_n)$  strictly interlace, and when  $\lambda_n > \lambda_k$  they lie strictly to the right of the zeros of  $L'(\lambda_0, \dots, \lambda_n, \dots, \lambda_k)$ ;*

Before getting into the proof, let us point out that in the proof of Theorem 4.5, on applying Theorem 4.6 repeatedly, we get

**Corollary 4.7.** *Let  $\lambda_0 = \mu_0 = 0$ , and  $\lambda_k > 0, \mu_k > 0, k = 1, 2, \dots, n$ . Then*

- (1) *if  $\max\{\lambda_1, \dots, \lambda_n\} \leq \min\{\mu_1, \dots, \mu_n\}$ , the the zeros of  $L'(\mu_0, \dots, \mu_n)$  lie to the right of  $L'(\lambda_0, \dots, \lambda_n)$ .*
- (2) *if  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n, 0 = \mu_0 < \mu_1 < \dots < \mu_n$ , and  $\mu_1 \geq \lambda_1, \dots, \mu_n \geq \lambda_n$ , then the zeros of  $L'(\mu_n, \dots, \mu_0)$  lie to the right of  $L'(\lambda_0, \dots, \lambda_n)$ .*

The proof of Theorem 4.6. depends on the following auxiliary Lemma, which specifies what kind of orthogonality the derivatives of Müntz–Legendre polynomials satisfy.

**Lemma 4.8.** *Let  $\{\lambda_0, \dots, \lambda_n\}$  satisfy (4.1). Then*

$$\int_0^1 L'(\lambda_0, \dots, \lambda_n; x)q(x)dx = 0$$

where  $q$  is in the linear span of  $\{1, x^{\lambda_0+1}, \dots, x^{\lambda_n+1}\}$  with  $q(0) = q(1) = 0$ .

*Proof.* This follows directly from integration by parts. Denoting  $L(\lambda_0, \dots, \lambda_n)$  by  $\Phi$ , then

$$\int_\epsilon^1 \Phi'(x)q(x)dx = \Phi(x)q(x)|_\epsilon^1 - \int_\epsilon^1 \Phi(x)q'(x)dx.$$

Letting  $\epsilon \rightarrow 0$ , the second term on the right tends to zero because of of orthogonality of  $\Phi$ , and because  $q'$  is in the linear span of  $\{x^{\lambda_0}, \dots, x^{\lambda_n-1}\}$ . The first term also tends to zero, because  $q(0) = q(1) = 0$ .  $\square$

*Proof of Theorem 4.6.* (i.) Let  $\Phi = L(\lambda_0, \dots, \lambda_{n-1})$  and  $\Psi = L(\lambda_0, \dots, \lambda_n)$ . Since  $\lambda_0 = 0$ , we see that  $\alpha\Phi' + \beta\Psi'$  is in  $\text{Span}\{x^{\lambda_1-1}, \dots, x^{\lambda_n-1}\}$ , which has at most  $n-1$  zeros in  $(0, 1)$ , namely,  $Z(\alpha\Phi' + \beta\Psi') \leq n-1$  when  $\alpha^2 + \beta^2 > 0$ . For the number of sign changes in  $(0, 1)$ , we have  $S^-(\alpha\Phi' + \beta\Psi') \geq n-2$ , since otherwise,  $S^-(\alpha\Phi' + \beta\Psi') \leq n-3$ , and we can choose a  $q$  in the linear span of  $\{1, x^{\lambda_0+1}, \dots, x^{\lambda_{n-2}}\}$ , such that  $q(0) = q(1) = 0$ , and  $q$  changes sign only when  $\alpha\Phi' + \beta\Psi'$  does. Without loss of generality, we can assume that  $(\alpha\Phi' + \beta\Psi')q > 0$  on  $(0, 1)$  except at finitely many points. This is a contradiction to Lemma 4.8. Hence we have  $n-2 \leq S^-(\alpha\Phi' + \beta\Psi') \leq Z(\alpha\Phi' + \beta\Psi') \leq n-1$ , and by Lemma 4.1, the zeros of  $\Phi'$  and  $\Psi'$  strictly interlace.

(ii.) Let  $\Phi = L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  and  $\Psi = L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n^*)$ . The interlacing property of zeros of  $\Phi'$  and  $\Psi'$  is proved in the same way as in (i) with the obvious modification:

$$n-1 \leq S^-(\alpha\Phi' + \beta\Psi') \leq Z(\alpha\Phi' + \beta\Psi') \leq n.$$

To see how the zeros move, we compare  $\Phi''(1)/\Phi'(1)$  and  $\Psi''(1)/\Psi'(1)$ . From Corollary 2.7, we get

$$\Phi''(1) = \lambda_n + S \quad \Psi''(1) = \lambda_n^* + S$$

where  $S = \sum_{k=0}^{n-1} (1 + 2\lambda_k)$ , and

$$\Phi''(1) = (\lambda_n - 1)(\lambda_n + S) + A, \quad \Psi''(1) = (\lambda_n^* - 1)(\lambda_n^* + S) + A,$$

with  $A = \sum_{k=0}^{n-1} (1 + 2\lambda_k) \left( \lambda_k + \sum_{j=0}^{k-1} (1 + 2\lambda_j) \right)$ . A calculation shows that

$$\frac{\Psi''(1)}{\Psi'(1)} - \frac{\Phi''(1)}{\Phi'(1)} = (\lambda_n^* - \lambda_n) \left[ 1 - \frac{A}{(\lambda_n + S)(\lambda_n^* + S)} \right].$$

Since  $\lambda_n^* > \lambda_n$ , and  $A < S^2$ , we conclude that  $\Psi''(1)/\Psi'(1) > \Phi''(1)/\Phi'(1)$ , which means that the zeros of  $\Psi'$  lie strictly to the right of  $\Phi'$ .

(iii.) Recall that in  $L(\lambda_0, \dots, \lambda_n)$ , a permutation of the first  $n$  indices (keeping the last index  $\lambda_n$  last) does not change the Müntz–Legendre polynomial, so we can assume without loss of generality that  $k = n - 1$ . Let  $\Phi = L(\lambda_0, \dots, \lambda_{n-1}, \lambda_n)$  and  $\Psi = L(\lambda_0, \dots, \lambda_n, \lambda_{n-1})$ . The interlacing property of zeros of  $\Phi'$  and  $\Psi'$  is proved in the same way as in (i) by showing that

$$n - 2 \leq S^-(\alpha\Phi' + \beta\Psi') \leq Z(\alpha\Phi' + \beta\Psi') \leq n - 1.$$

As for the lexicographical property of zeros of  $\Phi'$  and  $\Psi'$ , we apply Corollary 2.7 to get  $\Phi'(1) = \lambda_n + 1 + 2\lambda_{n-1} + S$ ,  $\Psi'(1) = \lambda_{n-1} + 1 + 2\lambda_n + S$ ,  $\Phi''(1) = (\lambda_n - 1)\Phi'(1) + (1 + 2\lambda_{n-1})(\lambda_n + S) + A$ , and  $\Psi''(1) = (\lambda_{n-1} - 1)\Psi'(1) + (1 + 2\lambda_n)(\lambda_n + S) + A$ , where  $S = \sum_{k=0}^{n-2} (1 + 2\lambda_k)$ , and  $A = \sum_{k=0}^{n-2} (1 + 2\lambda_k) \left( \lambda_k + \sum_{j=0}^{k-1} (1 + 2\lambda_j) \right)$ . Calculation shows that

$$\frac{\Psi''(1)}{\Psi'(1)} - \frac{\Phi''(1)}{\Phi'(1)} = \frac{(\lambda_{n-1} - \lambda_n)(A - S^2 - (\lambda_n + \lambda_{n-1})S - \lambda_n\lambda_{n-1})}{\Psi'(1)\Phi'(1)}$$

Since  $\lambda_n > \lambda_{n-1}$ ,  $A \leq S^2$ , and  $\Psi'(1)\Phi'(1) > 0$ , we have  $\Psi''(1)/\Psi'(1) > \Phi''(1)/\Phi'(1)$ , which implies that the zeros of  $\Psi'$  in  $(0, 1)$  lie strictly to the right of those of  $\Phi'$ .  $\square$

Finally in this section, we give rough and yet in some cases sharp estimate for the smallest and largest zeros of Müntz–Legendre polynomials via the help of Laguerre polynomials (cf. Corollary 2.3).

**Corollary 4.9.** *Let  $\lambda_0 > -1/2, \dots, \lambda_n > -1/2$ . Assume that  $x_1 < \dots < x_n$  are the zeros of  $L(\lambda_0, \dots, \lambda_n)$  in  $(0, 1)$ . Then,*

$$e^{-\frac{4n+2}{1+2\lambda_*}} < x_1 < \dots < x_n < e^{\frac{-j_1^2}{(1+2\lambda^*)(4n+2)}}, \quad (4.7)$$

where  $\lambda_* = \min\{\lambda_0, \dots, \lambda_n\}$ ,  $\lambda^* = \max\{\lambda_0, \dots, \lambda_n\}$  and  $j_1 > 3\pi/4$  is the smallest positive zero of the Bessel function  $J_0(z) = \sum_{k=0}^{\infty} (-z^2)^k / (k!2^k)^2$ .

*Proof.* Let  $\mathbf{L}_n$  be the  $n$ th Laguerre polynomial with respect to the weight  $e^{-x}$  on  $(0, \infty)$ , and let the zeros of  $\mathbf{L}_n$  be  $z_1 < \dots < z_n$ , then we have (cf. [Sze, p. 127–131])

$$\frac{j_1^2}{4} < z_1 < \dots < z_n < 4n + 2. \quad (4.8)$$

where the upper estimate is asymptotically sharp, and the lower estimate is sharp up to a constant (not exceeding  $4^4/(9\pi^2)$ ). Since  $n$  is fixed, we let  $\epsilon > 0$  be sufficiently small that  $\lambda_* - n\epsilon > -1/2$ . Then all the zeros of  $L(\lambda_0, \dots, \lambda_n)$  lie to the right of those of  $L(\lambda_*, \lambda_* - \epsilon, \dots, \lambda_* - n\epsilon)$  by Theorem 4.5(1). From the contour integral formula (2.5),  $L(\lambda_*, \lambda_* - \epsilon, \dots, \lambda_* - n\epsilon)$  tends to  $L(\lambda_*, \lambda_*, \dots, \lambda_*)$  uniformly on closed subintervals of  $(0, 1]$  as  $\epsilon \rightarrow 0$ . Recalling that (cf. Corollary 2.3)  $L(\lambda_*, \lambda_*, \dots, \lambda_*) = x^{\lambda_*} \mathbf{L}_n(-(1 + 2\lambda_*) \log x)$ , we conclude that  $x_1 \geq y_1$ , where  $y_1$  is the smallest zero of  $L(\lambda_*, \lambda_*, \dots, \lambda_*)$ . Since  $z_n = -(1 + 2\lambda_*) \log y_1$ , combining this with (4.8) we get

$$x_1 \geq y_1 = e^{\frac{-z_n}{1+2\lambda_*}} > e^{-\frac{4n+2}{1+2\lambda_*}},$$

which is the left half of (4.7). Similarly, we can claim that all zeros of  $L(\lambda_0, \dots, \lambda_n)$  lie to the left of zeros of  $L(\lambda^*, \dots, \lambda^*) = x^{\lambda^*} \mathbf{L}_n(-(1 + 2\lambda^*) \log x)$ , which implies that  $x_n < \exp(-j_1^2/(1 + 2\lambda^*)(4n + 2))$ .  $\square$

## §5. CHRISTOFFEL FUNCTIONS

Christoffel functions have been intensively studied, and their utility in orthogonal polynomials and approximation theory can be illustrated by their relation with polynomial inequalities, interpolation theory, quadrature formulas, zeros of orthogonal polynomials, etc (cf. [Nev3]). In this section, we will study some inequalities of Christoffel functions for Müntz–Legendre polynomial, and some of their applications.

We assume that  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  satisfies

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots \rightarrow +\infty \quad (5.1)$$

The Christoffel function for the Müntz system  $M(\Lambda)$  with respect to the Lebesgue weight is defined by either side of the following equality

$$\frac{1}{\sum_{k=0}^{\infty} |L_k^*(x)|^2} = \inf_{p \in M(\Lambda), p(x)=1} \int_0^1 |p(t)|^2 dt \quad (5.2)$$

which is a well known consequence of the orthogonality (cf. [Sze, p. 39]). If the infimum is taken just over  $M_n(\Lambda)$ , then we have

$$\frac{1}{\sum_{k=0}^n |L_k^*(x)|^2} = \min_{p \in M_n(\Lambda), p(x)=1} \int_0^1 |p(t)|^2 dt, \quad (5.3)$$

and either side can be called the  $n$ th Christoffel function. On recalling the reproducing kernel (2.20), we see that  $1/K(x, x)$ , and  $1/K_n(x, x)$  are what we just defined (cf. (3.2) and (3.3)). For convenience, we will defy the section title a little by stating results in terms of the reciprocal of the Christoffel functions, namely, in terms of  $K(x) = K(x, x)$  and  $K_n(x) = K_n(x, x)$ .

The classical Müntz theorem characterizes the denseness of  $M(\Lambda)$  by the divergence of the series  $\sum_{k=1}^{\infty} \lambda_k^{-1}$ . Now we can connect the Christoffel functions with the denseness

**Theorem 5.1.** *Let  $\Lambda = \{0 = \lambda_0 < \lambda_1 < \dots \rightarrow \infty\}$ . The the following are equivalent:*

- (1)  $M(\Lambda)$  is not dense in  $C[0, 1]$  in the uniform norm;
- (2)  $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$ ;
- (3) There is an  $x \in [0, 1)$ , such that  $\sum_{k=0}^{\infty} |L_k^*(x)|^2 < +\infty$ ;
- (4)  $\sum_{k=0}^{\infty} |L_k^*(x)|^2$  converges uniformly on  $[0, 1 - \epsilon]$  for every  $0 < \epsilon < 1$ .

The right endpoint 1 is quite different, where we always have (cf. (2.9) and (2.14))  $K(1) = \sum_{k=0}^{\infty} |L_k^*(1)|^2 = \sum_{k=0}^{\infty} (1 + 2\lambda_k) = +\infty$ . The following lemma is extracted from the proof of [ClEr, Theorem 3], see also [Bor, Lemma 2]. It estimates the function values and derivative values of  $\Lambda$ -polynomials on  $[0, 1 - \epsilon]$  by their  $L^2[0, 1]$  norms. The proof of Theorem 5.1 will follow this.

**Lemma 5.2.** *Let  $\Lambda = \{0 = \lambda_0 < \lambda_1 < \dots \rightarrow \infty\}$  be an integer index sequence with  $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$ . If  $\nu \geq 0$  is a fixed integer such that for  $k = 0, 1, \dots, \nu$ , either  $\lambda_k$  is an integer or  $\lambda_k \geq \nu$ , then*

$$\max_{x \in [0, 1 - \epsilon]} |p^{(\nu)}(x)| \leq C \left( \int_0^1 |p(x)|^2 \right)^{1/2}, \quad \text{for every } p \in M(\Lambda) \quad (5.4)$$

where  $0 < \epsilon < 1$ , and  $C = C(\Lambda, \epsilon, \nu)$  is a constant.

*Proof.* Since  $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$ , from the proof of [ClEr, Theorem 3], for every  $\epsilon > 0$ , there is constant  $C_0 = C_0(\Lambda, \epsilon) > 0$  such that for every  $\Lambda$ -polynomial  $p(x) = \sum_{k=0}^n a_k x^{\lambda_k}$  and every  $n$  that

$$|a_k| \leq C_0 (1 + \epsilon)^{\lambda_k} \left( \int_0^1 |p(x)|^2 dx \right)^{1/2}.$$

Note in particular that  $C_0$  is independent of  $n$  and  $p$ . Hence,

$$|p^{(\nu)}(x)| \leq \sum_{k=0}^n |a_k| \lambda_k^{\nu} x^{\lambda_k - \nu} \leq C_0 \sum_{k=0}^n (1 + \epsilon)^{\lambda_k} \left( \int_0^1 |p(x)|^2 dx \right)^{1/2} \lambda_k^{\nu} x^{\lambda_k - \nu}.$$

If  $x \in [0, 1 - \epsilon]$ , then  $(1 + \epsilon)x \leq 1 - \epsilon^2$ , and the above implies that

$$|p^{(\nu)}(x)| \leq C_0 (1 + \epsilon)^{\nu} \sum_{k=0}^{\infty} (1 - \epsilon^2)^k k^{\nu} \left( \int_0^1 |p(t)|^2 dt \right)^{1/2}.$$

Therefore, (5.4) holds with  $C(\Lambda, \epsilon, \nu) = C_0 (1 + \epsilon)^{\nu} \sum_{k=0}^{\infty} (1 - \epsilon^2)^k k^{\nu}$ .  $\square$

**Corollary 5.3.** *Under the condition of Lemma 5.2,*

$$\max_{x \in [0, 1-\epsilon]} |p(x)| \leq C \int_0^1 |p(x)| dx, \quad p \in M(\Lambda)$$

where  $C = C(\Lambda, \epsilon)$  is independent of  $p$ .

*Proof.* Consider the new Müntz system  $\{1, x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}$ , and apply Lemma 5.2 with  $\nu = 1$  for those Müntz polynomials  $\int_0^x p(t) dt$  with  $p \in M(\Lambda)$ , and use the simple fact that  $|\int_0^x p(t) dt| \leq \int_0^1 |p(t)| dt$ .  $\square$

*Proof of Theorem 5.1.* The equivalence of (i) and (ii) is the classical Müntz–Szász Theorem. We will follow (ii)  $\implies$  (iv)  $\implies$  (iii)  $\implies$  (i).

(ii)  $\implies$  (iv). Since  $\sum_{k=0}^{\infty} \lambda_k^{-1} < +\infty$ , we have by Theorem 3.1 that for every  $x \in [0, 1]$  that

$$\sum_{k=0}^{\infty} |L_k^{*(\nu)}(x)|^2 = \sup\{|p^{(\nu)}(x)|^2 : p \in M(\Lambda), \int_0^1 |p(x)|^2 dx = 1\}. \quad (5.5)$$

Hence by Lemma 5.2, for every  $\epsilon > 0$ , there is a constant  $C = C(\Lambda, \epsilon)$  such that

$$\sum_{k=0}^{\infty} |L_k^*(x)|^2 \leq C, \quad \sum_{k=0}^{\infty} |L_k^{*'}(x)|^2 \leq C. \quad (5.6)$$

Since

$$\frac{d}{dx} \sum_{k=0}^n (L_k^*(x))^2 = \sum_{k=0}^n L_k^*(x) L_k^{*'}(x), \quad (5.7)$$

on applying the Cauchy–Schwartz inequality and (5.6), we see that (5.7) is uniformly bounded by  $C$  for  $x \in [0, 1-\epsilon]$ , and  $n \geq 0$ . Therefore,  $\sum_{k=0}^n |L_k^*(x)|^2$  is equi-continuous for  $n = 0, 1, \dots$ , which implies the uniform convergence of  $K_n$  to  $K$  on  $[0, 1-\epsilon]$  by the Arzela–Ascoli Theorem.

(iv)  $\implies$  (iii) is trivial; We now finish the proof by demonstrating (iii)  $\implies$  (i). Assume for some  $x_0 \in [0, 1]$  that  $K(x_0) < +\infty$ . Then  $M(\Lambda)$  fails to be dense in  $C[0, 1]$ . Otherwise, let  $f \in C[0, 1]$ , such that  $|f(x_0)|^2 \geq K(x_0) + 2$ , and  $\int_0^1 |f(x)|^2 dx = 1$ . Then by the density assumption, there is a  $p \in M(\Lambda)$ , such that  $|p(x_0)|^2 \geq K(x_0) + 1$  and  $\int_0^1 |p(x)|^2 dx = 1$ , which means that  $\sup\{|p(x_0)| : p \in M(\Lambda), \int_0^1 |p(x)|^2 dx = 1\} \geq K(x_0) + 1$ . By (5.2)–(5.3) or Theorem 3.1, the above leads to  $K(x_0) \geq K(x_0) + 1$ , a contradiction.  $\square$

Actually when  $M(\Lambda)$  is not dense, the uniform convergence also holds for higher derivatives, and in this case, we don't require  $\lambda_k \rightarrow 0$ .

**Theorem 5.4.** *Let  $\Lambda = \{0 \leq \lambda_0 < \lambda_1 < \dots\}$  be a sequence of integers with  $\sum_{k=1}^{\infty} \lambda_k^{-1} < +\infty$ . Then*

$$\sum_{k=0}^{\infty} |L_k^{*(\nu)}(x)|^2 \quad \text{converges uniformly on } [0, 1 - \epsilon] \quad (5.8)$$

for every  $\nu = 0, 1, 2, \dots$  and every  $0 < \epsilon < 1$ .

*Proof.* The method is exactly the same as in the proof of (ii)  $\implies$  (iv) of Theorem 5.1. Using Lemma 5.2 to get the uniform boundedness of the series in (5.8) and the uniform boundedness of  $\sum_{k=0}^{\infty} |L_k^{*(\nu+1)}|^2$  on  $[0, 1 - \epsilon]$ , which implies the uniform boundedness of  $\frac{d}{dx} \sum_{k=0}^n |L_k^{*(\nu)}(x)|^2$  on  $[0, 1 - \epsilon]$  because of Cauchy-Schwarz inequality. And now the Arzela–Ascoli Theorem completes the proof.  $\square$

Immediately from Theorem 5.4, we have that the orthonormal Müntz–Legendre polynomials in the above situation tend to 0 uniformly on closed subintervals of  $[0, 1)$ . Whereas for orthogonal polynomials  $\{p_n\}$  orthonormal with respect to a measure supported on  $[0, 1]$ , only the relative growth  $|p_n|^2 / \sum_{k=0}^n |p_k|^2$  tends to 0 uniformly on  $[0, 1]$  (cf. [NTZ, NeZh, Zha]).

**Corollary 5.5.** *If  $0 = \lambda_0 < \lambda_1 < \dots \rightarrow \infty$ , and the associated Müntz system is not dense in  $C[0, 1]$ . Then*

$$\lim_{k \rightarrow \infty} \max_{x \in [0, 1 - \epsilon]} |L_k^{*(\nu)}(x)| = 0$$

holds for every  $0 < \epsilon < 1$  and every  $\nu = 0, 1, 2, \dots$ .

When the index sequence is lacunary, that is,

$$\inf\{\lambda_{k+1}/\lambda_k : k = 0, 1, 2, \dots\} > 1, \quad (5.9)$$

we can say more about the boundness of the function  $K$ . To do this, we first give a bounded Bernstein–type and a bounded Nikolskii–type inequality for a lacunary system (cf. [BoEr2, Theorem 3.1]).

**Lemma 5.6.** *Let  $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$  be lacunary as in (5.9). Then*

$$|p'(x)| \leq \frac{C}{1-x} \max_{t \in [0, 1]} |p(t)| \quad x \in [0, 1), p \in M(\Lambda), \quad (5.10)$$

and

$$|p(x)| \leq \frac{C}{1-x} \int_0^1 |p(t)| dt \quad x \in [0, 1), p \in M(\Lambda) \quad (5.11)$$

hold with the constant  $C = C(\Lambda)$  depending only on the system.

*Proof.* The inequality (5.10) comes from [BoEr2, Theorem 3.1]. For (5.11), consider the new lacunary sequence  $\Lambda^* = \{0, 1 + \lambda_0, 1 + \lambda_1, \dots\}$ , and apply (5.10) for those Müntz polynomials in  $M(\Lambda^*)$  which are indefinite integrals of  $p \in M(\Lambda)$ .  $\square$

**Theorem 5.7.** *Let  $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  be lacunary. Then there is a constant  $C = C(\Lambda)$ , such that*

$$K(x) = \sum_{k=0}^{\infty} |L_k^*(x)|^2 \leq \frac{C}{(1-x)^2}, \quad x \in [0, 1).$$

*Proof.* Since  $\Lambda$  is lacunary, applying Lemma 5.6, we get

$$|p(x)|^2 \leq \frac{C}{(1-x)^2} \left( \int_0^1 |p(t)| dt \right)^2 \leq \frac{C}{(1-x)^2} \int_0^1 |p(t)|^2 dt.$$

By (5.2)–(5.3) or (3.1)–(3.3), we have  $K(x) \leq C/(1-x)^2$ .  $\square$

As a last observation in this paper, we point out that there is a sequence  $x_n \rightarrow 1^-$ , such that  $K(x_n) \geq C_1/(1-x_n)$ . Indeed, let  $x_n = 1 - 1/\lambda_n$ , and consider  $p(x) = x^{\lambda_n}$ . Then by (5.2)–(5.3) or Theorem 3.1,  $K(x_n) \geq p(x_n)^2 / \|p\|_2^2 = x_n^{2\lambda_n} (2\lambda_n + 1) = (1 - 1/\lambda_n)^{2\lambda_n} (2\lambda_n + 1) \geq C_1 \lambda_n \geq C_1/(1-x_n)$ , where  $C_1 = \inf\{(1 - 1/\lambda_n)^{2\lambda_n} : n = 1, 2, 3, \dots\} > 0$ .

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