MARKOV-BERNSTEIN TYPE INEQUALITIES FOR POLYNOMIALS UNDER ERDŐS-TYPE CONSTRAINTS

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The Markov-Bernstein inequality asserts that

$$|p'(x)| \le \min\left\{\frac{n}{\sqrt{1-x^2}}\,,\, n^2\right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

holds for every polynomial of degree at most n with complex coefficients. Here, and in what follows, $\|p\|_A := \sup_{y\in A}|p(y)|$. Throughout his life Erdős showed a particular interest in inequalities for constrained polynomials. In a short paper in 1940 Erdős [7] has found a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(x)| \le \min\left\{\frac{c\sqrt{n}}{(1-x^2)^2}\,,\ \frac{en}{2}\right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for every polynomial p of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1,1)$. This result motivated a number of people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov-Bernstein type inequality of Erdős has been extended later in many directions.

Let $\mathcal{P}^c_{n,k}$ denote the set of all polynomials of degree at most n with *complex coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk. Let $\mathcal{P}_{n,k}$ denote the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk. Associated with $0 \le k \le n$ and $x \in (-1,1)$, let

$$B^*_{n,k,x} := \max\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\ n\log\left(\frac{e}{1-x^2}\right)\right\}, \qquad B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,,$$

and

$$M^*_{n,k} := \max\{n(k+1),\ n\log n\}\,, \qquad M_{n,k} := n(k+1)\,.$$

It is shown in [5] and [6] that

$$c_1 \min\{B^*_{n,k,x}, M^*_{n,k}\} \le \sup_{p\in\mathcal{P}^c_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 \min\{B^*_{n,k,x}, M^*_{n,k}\}$$

for every $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. This result should be compared with the inequalities

$$c_3 \min\{B_{n,k,x}, M_{n,k}\} \leq \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq c_4 \min\{B_{n,k,x}, M_{n,k}\}$$

for every $x \in (-1,1)$, where $c_3 > 0$ and $c_4 > 0$ are absolute constants. The upper bound of this second result is also fairly recent, see [1], and it may be surprising that there is a significant difference between the real and complex cases as far as Markov-Bernstein type inequalities are concerned. The lower bound of the second result is proved in [5]. It is the final piece of a long series of papers on this topic by a number of authors starting with Erdős in 1940.

Let $\mathcal{P}_n^c(r)$ be the set of all polynomials of degree at most n with *complex coefficients* and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$, respectively $(0 < r \leq 1)$. Let $\mathcal{P}_n(r)$ be the set of all polynomials of degree at most n with *real coefficients* and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$, respectively $(0 < r \leq 1)$.

Essentially sharp Markov-type inequalities for $\mathcal{P}_n^c(r)$ and $\mathcal{P}_n(r)$ on $[-1,1]$ are established in [6] and [4]. In [6] we show

$$c_1 \min\left\{\frac{n\log\left(e + n\sqrt{r}\right)}{\sqrt{r}}\,,\, n^2\right\} \leq \sup_{0 \neq p \in \mathcal{P}_n^c(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 \min\left\{\frac{n\log\left(e + n\sqrt{r}\right)}{\sqrt{r}}\,,\, n^2\right\}$$

for every $0 < r \leq 1$ with absolute constants $c_1 > 0$ and $c_2 > 0$. This result should be compared with the inequalities

$$c_3 \min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\} \leq \sup_{0 \neq p \in \mathcal{P}_n(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_4 \min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}, \qquad 0 < r \leq 1\,,$$

where $c_3 > 0$ and $c_4 > 0$ are absolute constants. See [4].

Let K_α be the open diamond of the complex plane with diagonals $[-1,1]$ and $[-ia, ia]$ such that the angle between $[ia, 1]$ and $[1, -ia]$ is $\alpha\pi$. In [8] Halász proved that there are constants $c_1 > 0$ and $c_2 > 0$ depending only on α such that

$$c_1 n^{2-\alpha} \leq \sup_p \frac{|p'(1)|}{\|p\|_{[-1,1]}} \leq \sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 n^{2-\alpha}\,,$$

where the supremum is taken for all polynomials p of degree at most n (with either real or complex coefficients) having no zeros in K_α.

Erdős had many questions and results about polynomials with restricted coefficients. Let $\mathcal{F}_n$ denote the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$. Let $\mathcal{G}_n$ be the collection of polynomials p of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_m| = 1\,, \quad |a_j| \leq 1\,,$$

where m is an unspecified nonnegative integer not greater than n. In [2] and [3] we established the right Markov-type inequalities for the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ on $[0,1]$. Namely there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{F}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n \log(n+1)$$

and

$$c_1 n^{3/2} \leq \max_{0 \neq p \in \mathcal{G}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n^{3/2}\,.$$

Observe that the right Markov factor for $\mathcal{G}_n$ is much larger than the right Markov factor for $\mathcal{F}_n$. We also show that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \log(n+1) \leq \max_{0 \neq p \in \mathcal{L}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \leq c_2 n \log(n+1)\,,$$

where $\mathcal{L}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1,1\}$.

For polynomials

$$p \in \mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n \quad \text{with} \quad |p(0)| = 1$$

and for $y \in [0,1)$ the Bernstein-type inequality

$$\frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} \leq \max_{\substack{p \in \mathcal{F} \\ |p(0)|=1}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \leq \frac{c_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

is also proved with absolute constants $c_1 > 0$ and $c_2 > 0$.

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